21-241 Fall 03 Sec 3

Final Exam Review Sheet Solns

Note Title 12/5/2003

1. a) 4 eqns, 3 unknowns so some variable is free.

   b) You know how to do this by now!

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2. $$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & 3 & 4 \\ 4 & 5 & 4 \end{bmatrix} \xrightarrow[\substack{R_3 - 3R_1 \\ R_4 - 4R_1}]{R_2 - 2R_1} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$\xrightarrow[\text{row 2}]{\text{swap row 4 with}} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

3 Pivots $\Longrightarrow$ Linearly independent

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3. $f\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. $f\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$.

   So $f\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$.

4. a) $A = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

b) eigenvalues of A are $\left\{ \frac{3+\sqrt{5}}{2}, 1, \frac{3-\sqrt{5}}{2} \right\}$

Corresponding eijenvecs: $v_1 = \begin{bmatrix} \frac{3+\sqrt{5}}{2} - 1 \\ 1 \\ 0 \end{bmatrix}, v_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, v_3 = \begin{bmatrix} \frac{3-\sqrt{5}}{2} - 1 \\ 1 \\ 0 \end{bmatrix}$

By Normalizig these evecs, we have

$$P = \begin{bmatrix} | & | & | \\ \frac{v_1}{\|v_1\|}, & \frac{v_2}{\|v_2\|}, & \frac{v_3}{\|v_3\|} \\ | & | & | \end{bmatrix}.$$

and $AP = P \underbrace{\begin{bmatrix} \frac{3+\sqrt{5}}{2} & & \\ & 1 & \\ & & \frac{3-\sqrt{5}}{2} \end{bmatrix}}_{\text{call this } D}$.

So the O.Diag. is $A = PDP^T$.

c) All eigenvalues $> 0 \Rightarrow$ A positive definite

$\Longrightarrow f(x,y,z) \geq 0$ and

$f(x,y,z) = 0 \Leftrightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

d) $\max\limits_{x : \|x\| = 1} x^T A x = \lambda_1 = \frac{3+\sqrt{5}}{2}$.

$\min\limits_{x : \|x\| = 1} x^T A x = \lambda_3 = \frac{3-\sqrt{5}}{2}$.