21-241 Fall 03 Sec C

Exam 3 Review Sheet Solns

Note Title 11/22/2003

1. If 2 matricies are diagonalizable and produce the same diagonal matrix when diagonalized then they are similar. (See why?)

Now $\begin{bmatrix} 1 & 0 & -1 \\ 0 & 2 & -2 \\ 0 & 0 & 3 \end{bmatrix}$ has eigenvalues $1, 2, 3$. So it is diagonalizable (since the eigenvalues are distinct), so to show it is similar to $\begin{bmatrix} 3 & 1 & 1 \\ -2 & 0 & -3 \\ 0 & 0 & 3 \end{bmatrix}$, it is sufficient to show this has eigenvalues $\{1, 2, 3\}$

$$\det\left(\begin{bmatrix} 3 & 1 & 1 \\ -2 & 0 & -3 \\ 0 & 0 & 3 \end{bmatrix} - I\right) = 2\begin{vmatrix} 2 & 1 \\ -2 & -1 \end{vmatrix} = 0$$

$$\det\left(\begin{bmatrix} 3 & 1 & 1 \\ -2 & 0 & -3 \\ 0 & 0 & 3 \end{bmatrix} - 2I\right) = 1\begin{vmatrix} 1 & 1 \\ -2 & -2 \end{vmatrix} = 0$$

$$\det\left(\begin{bmatrix} 3 & 1 & 1 \\ -2 & 0 & -3 \\ 0 & 0 & 3 \end{bmatrix} - 3I\right) = 0\begin{vmatrix} 0 & 1 \\ -2 & -3 \end{vmatrix} = 0$$

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2. $\det\left(\begin{bmatrix} 4 & 3 \\ -3 & 4 \end{bmatrix} - \lambda I\right) = (4-\lambda)^2 + 9$

$$= 16 - 8\lambda + \lambda^2 + 9 = \lambda^2 - 8\lambda + 25$$

Setting this equal to 0, we find

$$\lambda = \frac{8 \pm \sqrt{64-100}}{2} = 4 \pm 3i.$$

$$E_{4+3i} = \mathrm{Nul}\begin{bmatrix} -3i & 3 \\ -3 & -3i \end{bmatrix} = \mathrm{span}\left\{\begin{bmatrix} 1 \\ i \end{bmatrix}\right\}$$

$$E_{4-3i} = \mathrm{Nul}\begin{bmatrix} 3i & 3 \\ -3 & 3i \end{bmatrix} = \mathrm{span}\left\{\begin{bmatrix} 1 \\ -i \end{bmatrix}\right\}$$

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3. $\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}$ has 2 distinct eigenvalues, 1 & 2, so we must determine the dimensions of the eigenspaces:

$$E_2 = \mathrm{Nul}\begin{bmatrix} -1 & 0 & 0 \\ 1 & -1 & 0 \\ 1 & 1 & 0 \end{bmatrix} = \mathrm{Nul}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \mathrm{span}\left\{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right\}$$

$$E_1 = \mathrm{Nul}\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix} = \mathrm{Nul}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \mathrm{span}\left\{\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}\right\}$$

$\dim E_2 + \dim E_1 = 1 + 1 < 3 \Rightarrow$ Not diagonalizable.

4. If A is diagonalizable, then $A = P^{-1}DP$ and $A^n = P^{-1}D^nP$.

So just diagonalize $\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$ and you've got it.

Unfortunately, this is a bit of a pain... the end result is:

$$\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} = \left(\frac{1}{\sqrt{5}}\begin{bmatrix} 1 & 1-\sqrt{5} \\ -1 & 1+\sqrt{5} \end{bmatrix}\right)\begin{bmatrix} \frac{1+\sqrt{5}}{2} & 0 \\ 0 & \frac{1-\sqrt{5}}{2} \end{bmatrix}\begin{bmatrix} \frac{1+\sqrt{5}}{2} & \frac{1-\sqrt{5}}{2} \\ 1 & 1 \end{bmatrix}$$

So

$$\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^n = \left(\frac{1}{\sqrt{5}}\begin{bmatrix} 1 & 1-\sqrt{5} \\ -1 & 1+\sqrt{5} \end{bmatrix}\right)\begin{bmatrix} \left(\frac{1+\sqrt{5}}{2}\right)^n & 0 \\ 0 & \left(\frac{1-\sqrt{5}}{2}\right)^n \end{bmatrix}\begin{bmatrix} \frac{1+\sqrt{5}}{2} & \frac{1-\sqrt{5}}{2} \\ 1 & 1 \end{bmatrix}$$

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5. $\text{Row}\,A = \text{Span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \\ 1 \end{bmatrix} \right\}$.

To find an orthonormal basis, we apply the Gram-Schmidt process:

$$v_1 = \begin{bmatrix} 1 \\ 1 \\ 2 \\ 1 \end{bmatrix}. \quad v_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} - \frac{5}{7}\begin{bmatrix} 1 \\ 1 \\ 2 \\ 1 \end{bmatrix} = \frac{1}{7}\begin{bmatrix} 2 \\ 2 \\ -3 \\ 2 \end{bmatrix}.$$

$$v_3 = \begin{bmatrix} 1 \\ -1 \\ 1 \\ 1 \end{bmatrix} - \frac{3}{7}\begin{bmatrix} 1 \\ 1 \\ 2 \\ 1 \end{bmatrix} - \frac{-1/7}{21/49}\left(\frac{1}{7}\begin{bmatrix} 2 \\ 2 \\ -3 \\ 2 \end{bmatrix}\right) = \cdots$$

6. U is orthonormal, so if $U = \begin{bmatrix} | & | & & | \\ u_1 & u_2 & \cdots & u_n \\ | & | & & | \end{bmatrix}$

then $\{u_1, u_2, \ldots, u_n\}$ is orthonormal.

So $U^T U = \begin{bmatrix} — u_1 — \\ \vdots \\ — u_n — \end{bmatrix} \begin{bmatrix} | & & | \\ u_1 & \cdots & u_n \\ | & & | \end{bmatrix}$

$$= \begin{bmatrix} u_1 \cdot u_1 & u_1 \cdot u_2 & \cdots & u_1 \cdot u_n \\ u_2 \cdot u_1 & u_2 \cdot u_2 & \cdots & u_2 \cdot u_n \\ \vdots & & & \vdots \\ u_n \cdot u_1 & u_n \cdot u_2 & \cdots & u_n \cdot u_n \end{bmatrix} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ 0 & 0 & \cdots & 1 \end{bmatrix}$$

So (by I.M.T.) $U^T = U^{-1}$

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7. We do not want to write down the characteristic polynomial of this beast!

"Note" that (using the "theoretical" definition of matrix multiplication),

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 2 & 2 \\ 3 & 3 & 3 & 3 & 3 \\ 4 & 4 & 4 & 4 & 4 \\ 5 & 5 & 5 & 5 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{bmatrix} + \cdots + x_5 \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{bmatrix}$$

$$= (x_1 + x_2 + x_3 + x_4 + x_5) \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{bmatrix}.$$

If $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{bmatrix}$, then we have an eigenvalue

$\lambda_1 = 15$.

Also note that the matrix is not invertible, so there is an eigenvalue $\lambda_2 = 0$.

Now the Null space of matrix $= E_0$

$$= \mathrm{Nul}\left(\begin{bmatrix} 1&1&1&1&1 \\ 0&0&0&0&0 \\ 0&0&0&0&0 \\ 0&0&0&0&0 \\ 0&0&0&0&0 \end{bmatrix}\right) = \mathrm{span}\left\{\begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ -1 \end{bmatrix}\right\}$$

Since $\dim E_0 + \dim E_{15} \leq 5$ and

$\dim E_0 = 4$, we must have $\dim E_{15} = 1$

and there is no room for other eigenvalues

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8. If eigenvalues of A are $\lambda_1, \lambda_2, \dots, \lambda_n$, then

for each λ_i, there exists v_i with

$$A v_i = \lambda_i v_i.$$

So $\frac{1}{\lambda_i} v_i = A^{-1} v_i$, and $\frac{1}{\lambda_i}$ is an eigenvalue of A^{-1}.

Similarly, the eigenvalues of $A+I$ are $(\lambda_1+1), (\lambda_2+1), \dots, (\lambda_n+1)$,

and the eigenvalues of $(A+I)^{-1}$ are $\frac{1}{\lambda_1+1}, \frac{1}{\lambda_2+1}, \dots, \frac{1}{\lambda_n+1}$.

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9. The $\vec{x}$ we want is

$$\vec{x} = \mathrm{Proj}_{\mathrm{span}\{\vec{b},\vec{c}\}} \vec{a} \,. \quad \text{(see why?)}$$

Unfortunately $\vec{b}$ is not orthogonal to $\vec{c}$, so we must find an orthogonal basis to project.

$$v_1 = \vec{b}. \qquad v_2 = \vec{c} - \frac{\vec{b}\cdot\vec{c}}{\vec{b}\cdot\vec{b}}\vec{b}.$$

Then $$\vec{x} = \frac{\vec{a}\cdot v_1}{v_1\cdot v_1}v_1 + \frac{\vec{a}\cdot v_2}{v_2\cdot v_2}v_2 .$$

(I don't feel like doing the arithmetic... you do it.)

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10. How about: $\left\{ \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix} \right\}$

There are plenty of other answers, but 3 indep vecs in

$\mathbb{R}^3$ will have $\text{span}\{v_1, v_2, v_3\} = \mathbb{R}^3$, always.

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11. Hmm, I don't think this true...

If $x = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$, $y = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$, $x_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$, $x_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$

Then

$x = x_1 + x_2$, $x_2 \in L^\perp$, $x_1 \cdot x_2 = 0$;

but $x_1 \neq \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$. My bad.

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12. $x^T A^T A x = (Ax) \cdot (Ax) \geq 0$,

since $y \cdot y \geq 0$ for any vector y.

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13. We will show that the columns of A are linearly independent. (Why is this sufficient?)

Suppose $c_1 \vec{a_1} + c_2 \vec{a_2} + \cdots + c_n \vec{a_n} = \vec{0}$.

Then $\vec{a_i} \cdot (c_1 \vec{a_1} + \cdots + c_n \vec{a_n}) = \vec{a_i} \cdot \vec{0} = 0$

$= 0 + \cdots + 0 + c_i + 0 + \cdots + 0$ so $c_i = 0$. ∎