21-241 Sec C Exam 2 Review Sheet Solutions

---

① You know how to rotate 45° about the origin still? Send $(1,0)$ to $(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$ and $(0,1)$ to $(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$. In homogeneous coordinates this is represented by
$$\begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

To rotate 45° about the point $(51,-7)$ you can

① Translate $(51,-7)$ to the origin

② Rotate 45° about the origin

③ Translate the origin back to $(51,-7)$

How do you translate $(51,-7)$ to the origin?

Ans: $\begin{bmatrix} 1 & 0 & -51 \\ 0 & 1 & 7 \\ 0 & 0 & 6 \end{bmatrix}$

How do you translate the origin to $(51,-7)$?

To represent ① followed by ② followed by ③ as a single ~~matrix~~, we multiply:

$$\begin{bmatrix} 1 & 0 & 51 \\ 0 & 1 & -7 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -51 \\ 0 & 1 & 7 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 51 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & -7 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -51 \\ 0 & 1 & 7 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & -\frac{58}{\sqrt{2}}+51 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & -\frac{44}{\sqrt{2}}-7 \\ 0 & 0 & 1 \end{bmatrix}$$

---

② $\left[\begin{array}{cc|cc} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow[R_1 \leftrightarrow R_2]{\text{swap}} \left[\begin{array}{cc|cc} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{array}\right]$. So $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}^{-1} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

$$\left[\begin{array}{ccc|ccc} 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow{\text{Replace } R_1 \text{ w/ } \frac{1}{2}(R_1+R_2+R_3)} \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\longrightarrow \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 0 & -1 & 0 & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & -1 & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \end{array}\right] \longrightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 0 & 1 & 0 & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 1 & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \end{array}\right]$$

Do you have a guess for the $n \times n$ case based on the 3×3 case?

$$\left[\begin{array}{cccccc|cccccc} 0 & 1 & 1 & \cdots & 1 & 1 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & 1 & \cdots & 1 & 1 & 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & & & & & & & & & \ddots & & \\ 1 & 1 & 1 & \cdots & 1 & 0 & 0 & 0 & 0 & \cdots & 0 & 1 \end{array}\right] \to \left[\begin{array}{cccccc|cccccc} 1 & 1 & 1 & \cdots & 1 & 1 & \frac{1}{n-1} & & \cdots & & & \frac{1}{n-1} \\ 1 & 0 & 1 & \cdots & 1 & 1 & 0 & 1 & 0 & \cdots & 0 & 0 \\ & & & \ddots & & & & & & \ddots & & \\ 1 & 1 & 1 & \cdots & 1 & 1 & 0 & 0 & 0 & \cdots & 0 & 1 \end{array}\right]$$

$$\to \left[\begin{array}{cccccc|cccc} 1 & 1 & 1 & \cdots & 1 & 1 & \frac{1}{n-1} & & \cdots & \frac{1}{n-1} \\ 0 & -1 & 0 & \cdots & 0 & 0 & -\frac{1}{n-1} & \frac{n-2}{n-1} & \cdots & -\frac{1}{n-1} \\ & & & \ddots & & & & & \ddots & \\ 0 & 0 & 0 & \cdots & 0 & -1 & -\frac{1}{n-1} & & \cdots & \frac{n-2}{n-1} \end{array}\right]$$

$$\to \left[\begin{array}{cccc|cccc} 1 & 0 & \cdots & 0 & -\frac{n-2}{n-1} & \frac{1}{n-1} & \cdots & \frac{1}{n-1} \\ 0 & 1 & \cdots & 0 & \frac{1}{n-1} & -\frac{n-2}{n-1} & \cdots & \frac{1}{n-1} \\ & & \ddots & & & & \ddots & \\ 0 & 0 & \cdots & 1 & \frac{1}{n-1} & \frac{1}{n-1} & \cdots & -\frac{n-2}{n-1} \end{array}\right]$$

3

$$\left[\begin{array}{cccc|cccc} 3 & 3 & 3 & 1 & 1 & 0 & 0 & 0 \\ 3 & 1 & 0 & 3 & 0 & 1 & 0 & 0 \\ -3 & 0 & 1 & 3 & 0 & 0 & 1 & 0 \\ 0 & 3 & 3 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{cccc|cccc} 3 & 0 & 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 3 & -1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 3 & 1 & 0 & 1 & -1 \\ 0 & 3 & 3 & 1 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\to \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & \frac{1}{3} & 0 & 0 & -\frac{1}{3} \\ 0 & 1 & 0 & 3 & -1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 3 & 1 & 0 & 1 & -1 \\ 0 & 0 & 0 & -17 & 0 & -3 & -3 & 1 \end{array}\right]$$

$$\longrightarrow \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1/3 & 0 & 0 & -1/3 \\ 0 & 1 & 0 & 0 & -1 & 8/17 & -9/17 & 20/17 \\ 0 & 0 & 1 & 0 & 1 & -9/17 & 8/17 & -14/17 \\ 0 & 0 & 0 & 1 & 0 & 3/17 & 3/17 & -1/17 \end{array}\right]$$

Out[3]//MatrixForm=

So $A^{-1} = \begin{pmatrix} \frac{1}{3} & 0 & 0 & -\frac{1}{3} \\ -1 & \frac{8}{17} & -\frac{9}{17} & \frac{20}{17} \\ 1 & -\frac{9}{17} & \frac{8}{17} & -\frac{14}{17} \\ 0 & \frac{3}{17} & \frac{3}{17} & -\frac{1}{17} \end{pmatrix}$

---

(4) Since AB is invertible, there exists C s.t. $(AB)C = I$. But then $A(BC) = I$. Exhibiting a matrix D with $AD = I$ is sufficient to show A is invertible, so we take $D = BC$ and we are done. $\blacksquare$

---

(5) Here is 1 possible soln:

$A = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$. Then $\operatorname{col} A = \operatorname{span}\left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \end{bmatrix}\right\}$

$= \operatorname{span}\left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\}$.

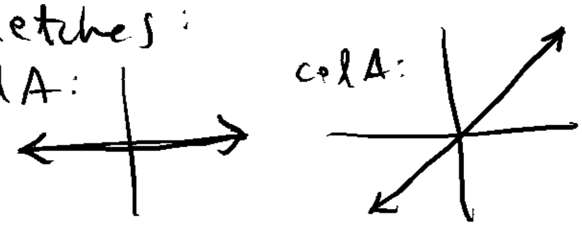
$\operatorname{nul} A = \{\vec{x} : A\vec{x} = \vec{0}\} = \left\{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} : x_1 = 0,\ x_2 \in \mathbb{R}\right\}$

$= \operatorname{span}\left\{\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right\}$.

Sketches:

nul A:

col A:



(6) Row Reduce:

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \\ -1 & 1 & 1 & -1 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & -2 & -2 \\ 0 & -2 & -2 & 0 \\ 0 & 2 & 2 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & -2 & -2 & 0 \\ 0 & 0 & -2 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

So a basis for null space is $\left\{ \begin{bmatrix} -1 \\ 1 \\ -1 \\ 1 \end{bmatrix} \right\}$.

And basis for column space is $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix} \right\}$.

If we had removed column 3 and row reduced, we would still find 3 pivot columns. So $\begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}$ is another basis for column space.

Since it doesn't change the span to multiply a basis vector by a nonzero constant, we have that

$$\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ -2 \\ 2 \\ -2 \end{bmatrix} \right\}$$

is a basis for the column space containing vector $\begin{bmatrix} 2 \\ -2 \\ 2 \\ -2 \end{bmatrix}$.

(7) $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$.

$A^2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $B^2 = \begin{bmatrix} 0 & -2 \\ 2 & 0 \end{bmatrix}$ so $A^2 - B^2 = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}$.

$A+B = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$, $A-B = \begin{bmatrix} -1 & 2 \\ 0 & -1 \end{bmatrix}$ so $(A+B)(A-B)$

$= \begin{bmatrix} -1 & 2 \\ -2 & 3 \end{bmatrix}$.

---

(8) Since $\dim \operatorname{nul} A = 3$ and $\dim \operatorname{nul} A + \dim \operatorname{col} A = 3$, $\dim \operatorname{col} A = 0$. So $\operatorname{col} A = \{\vec{0}\}$ (the trivial subspace). The only matrix A with this property is

$$A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

---

(9) $\dim \operatorname{col} A = 3$. $\dim \operatorname{col} A + \dim \operatorname{Nul} A = 3$. So $\operatorname{nul} A = \{\vec{0}\}$. Now any Invertible matrix A has this property.

---

(10) Yes. Some sequence of Row Operations transformed A to A'. If we apply the same sequence of Row Ops to just columns 2 + 4 of A, we get $\begin{bmatrix} 9 & -7 \\ -6 & 8 \\ -9 & 2 \end{bmatrix} \longrightarrow \begin{bmatrix} -3 & 9 \\ 0 & 5 \\ 0 & 0 \end{bmatrix}$. This has 2 pivots

So $\begin{bmatrix} 9 \\ -6 \\ -9 \end{bmatrix}$ and $\begin{bmatrix} -7 \\ 8 \\ 2 \end{bmatrix}$ are linearly independent.

Since col A is 2-dimensional, these 2 vectors must span it.

---

(11) $\begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = 0 \cdot 0 - 1 \cdot 1 = -1$. $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ has 2 identical rows, so det $= 0$.

$$\begin{vmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{vmatrix} = -1 \begin{vmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{vmatrix} = (-1)(1) \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = 1.$$

---

(12) $$\det(A^3 A^T A^{-1}) = (\det A)^3 \det(A^T) \det(A^{-1})$$
$$= (\det A)^3 \det A \frac{1}{\det A}$$
$$= (\det A)^3 = 3^3 = 27.$$

---

(13) Here is the simple example:

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}.$$

Quick check:
$$\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 0 & 0 \\ 0 & 3-\lambda & 0 \\ 0 & 0 & 3-\lambda \end{vmatrix}$$
$$= (1-\lambda)(3-\lambda)^2.$$

$$E_1 = \operatorname{span}\left\{\begin{bmatrix}1\\0\\0\end{bmatrix}\right\}, \quad E_3 = \operatorname{span}\left\{\begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}\right\}$$

So some wild eigenvectors are

$$\begin{bmatrix}(10^{10})^{10}\\0\\0\end{bmatrix} \quad \text{(corresponding to eigenvalue 1)}$$

and

$$\begin{bmatrix}0\\\sqrt{2\pi}\\\sum_{i=0}^{\infty}\frac{1}{10^i}\end{bmatrix} \quad \text{(corresponding to eigenvalue 3)}$$

---

(14) To show 3 is an eigenvalue, we could evaluate the characteristic polynomial at $\lambda = 3$:

$$\det(A - 3I) = \begin{vmatrix}-3 & 1 & 1 & 1\\1 & -3 & 1 & 1\\1 & 1 & -3 & 1\\1 & 1 & 1 & -3\end{vmatrix} = \_\_\_ .$$

But in this case, it is easier to say what does A do to vector $\begin{bmatrix}1\\1\\1\\1\end{bmatrix}$?

$$\begin{bmatrix}0&1&1&1\\1&0&1&1\\1&1&0&1\\1&1&1&0\end{bmatrix}\begin{bmatrix}1\\1\\1\\1\end{bmatrix} = \begin{bmatrix}3\\3\\3\\3\end{bmatrix} = 3\begin{bmatrix}1\\1\\1\\1\end{bmatrix}.$$

So 3 is an eigenvalue (and $\begin{bmatrix}1\\1\\1\\1\end{bmatrix}$ is a corresponding eigenvector).

To show $\begin{bmatrix}-1\\1\\0\\0\end{bmatrix}$ is an eigenvector, we just multiply.

$$\begin{bmatrix}0&1&1&1\\1&0&1&1\\1&1&0&1\\1&1&1&0\end{bmatrix}\begin{bmatrix}-1\\1\\0\\0\end{bmatrix}=\begin{bmatrix}1\\-1\\0\\0\end{bmatrix}=(-1)\begin{bmatrix}-1\\1\\0\\0\end{bmatrix}.$$

This "shows" $\begin{bmatrix}-1\\1\\0\\0\end{bmatrix}$ is an eigenvector, corresponding to eigenvalue -1.