

Multiple Petersen subdivisions in permutation graphs*

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Abstract

A permutation graph is a cubic graph admitting a 1-factor M whose complement consists of two chordless cycles. Extending results of Ellingham and of Goldwasser and Zhang, we prove that if e is an edge of M such that every 4-cycle containing an edge of M contains e , then e is contained in a subdivision of the Petersen graph of a special type. In particular, if the graph is cyclically 5-edge-connected, then every edge of M is contained in such a subdivision. Our proof is based on a characterization of cographs in terms of twin vertices. We infer a linear lower bound on the number of Petersen subdivisions in a permutation graph with no 4-cycles, and give a construction showing that this lower bound is tight up to a constant factor.

1 Introduction

A special case of Tutte's 4-flow conjecture [7] states that every bridgeless cubic graph with no minor isomorphic to the Petersen graph is 3-edge-colourable. Before this special case was shown to be true by Robertson et al. (cf. [6]), one of the classes of cubic graphs for which the conjecture was known to hold was the class of permutation graphs — i.e., graphs with a 2-factor consisting of two

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chordless cycles. Indeed, by a result of Ellingham [3], every permutation graph is either Hamiltonian — and hence 3-edge-colourable — or contains a subdivision of the Petersen graph. To state his theorem more precisely, we introduce some terminology.

Rephrasing the above definition, a cubic graph G is a *permutation graph* if it contains a perfect matching M such that $G - E(M)$ is the disjoint union of two cycles, none of which has a chord in G . A perfect matching M with this property is called a *distinguished matching* in G . For brevity, if G is a permutation graph with a distinguished matching M , then the pair (G, M) is referred to as a *marked permutation graph*.

We let P_{10} be the Petersen graph. Given a distinguished matching M in G , an *M -copy of P_{10}* is a subgraph G' of G isomorphic to a subdivision of P_{10} and composed of the two cycles of $G - E(M)$ together with five edges of M . Following Goldwasser and Zhang [4], an *M -copy of P_{10}* is also referred to as an *M - P_{10}* . Furthermore, an *M -copy of the 4-cycle C_4* (or an *M - C_4*) is a 4-cycle in G using two edges of M .

The proof of Ellingham's result implies that if a marked permutation graph (G, M) contains no M - P_{10} , then it contains an M - C_4 (and is therefore Hamiltonian). Goldwasser and Zhang [4] obtained a slight strengthening:

Theorem 1. *If (G, M) is a marked permutation graph, then G contains either two M -copies of C_4 , or an M -copy of P_{10} .*

Lai and Zhang [5] studied permutation graphs satisfying a certain minimality condition and proved that in a sense, they contain 'many' subdivisions of the Petersen graph.

The main result of this note is the following generalization of Theorem 1.

Theorem 2. *Let (G, M) be a marked permutation graph on at least six vertices and let $e \in E(M)$. If e is contained in every M - C_4 of G , then e is contained in an M -copy of P_{10} .*

Theorem 2 is established in Section 2. The proof is based on a relation between M -copies of P_{10} in permutation graphs and induced paths in a related class of graphs.

Of particular interest is the corollary for cyclically 5-edge-connected graphs, that is, graphs containing no edge-cut of size at most 4 whose removal leaves at least two non-tree components.

Corollary 3. *Every edge of a cyclically 5-edge-connected marked permutation graph (G, M) is contained in an M - P_{10} .*

The class of cyclically 5-edge-connected permutation graphs is richer than one might expect. Indeed, it had been conjectured [8] that every cyclically 5-edge-connected permutation graph is 3-edge-colourable, but this conjecture has been recently disproved [2, Observation 4.2].

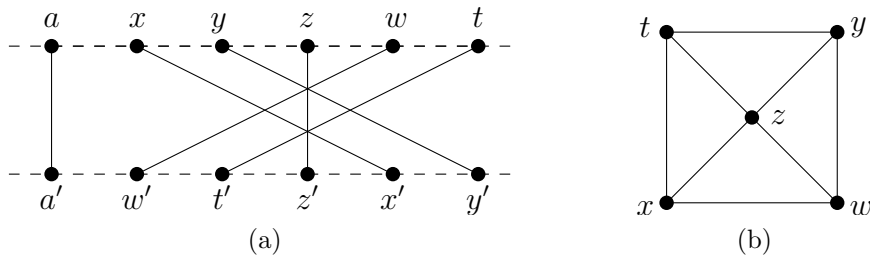


Figure 1: (a) The standard drawing of the graph G . (b) The corresponding graph H_a .

Theorem 2 readily implies a lower bound on the number of M -copies of the Petersen graph in a marked permutation graph (G, M) such that G contains no M - C_4 and has n vertices. We improve this lower bound in Section 3. We also show that the bounds (which are linear in n) are optimal up to a constant factor.

We close this section with some terminology. If G is a graph and $X \subseteq V(G)$, then $G[X]$ is the induced subgraph of G on X . The set of all neighbours of a vertex v of G is denoted by $N_G(v)$.

2 Proof of Theorem 2

Let (G, M) be a marked permutation graph. If $v \in V(G)$, then we write v' for the neighbour of v in M (which we call the *friend* of v). We extend this notation to arbitrary sets of vertices of G : if $X \subseteq V(G)$, then we set

$$X' = \{v' : v \in X\}.$$

Let A be the vertex set of one component of $G - E(M)$. Thus, A' is the vertex set of the other component and both $G[A]$ and $G[A']$ are chordless cycles.

In this section, we prove Theorem 2. Fix an edge e of the matching M . Let a and a' be its end-vertices. We also choose an orientation for each of the cycles $G[A]$ and $G[A']$. All these will be fixed throughout this section.

If $X \subseteq A$, then $G\{X\}$ is the spanning subgraph of G obtained by adding to $G - E(M)$ all the edges vv' , where $v \in X$. In expressions such as $G\{\{a, b\}\}$, we omit one pair of set brackets, and write just $G\{a, b\}$.

The *auxiliary graph* H_a (with respect to the vertex a) is defined as follows. The vertex set of H_a is $A - \{a\}$. Two vertices x and y of H_a are adjacent in H_a whenever the cyclic order of a, x and y on $G[A]$ is axy and the cyclic order of their friends on $G[A']$ is $a'y'x'$.

Alternatively, consider the following standard procedure, illustrated in Figure 1. Arrange the vertices of A on a horizontal line in the plane, starting on the left with a and continuing along the cycle $G[A]$ according to the fixed orientation. Place the vertices of A' on another horizontal line, putting a' leftmost and

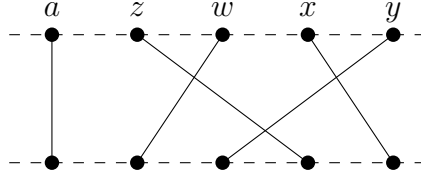


Figure 2: The graph $G\{a, x, y, z, w\}$ in Case 1 of the proof of Lemma 4.

continuing in accordance with the orientation of $G[A']$. Join each vertex $z \in A$ with its friend by a straight line segment. The segment aa' is not crossed by any other segment, and for $x, y \in A - \{a\}$, the segments xx' and yy' cross each other if and only if x and y are adjacent in H_a . Thus, H_a can be directly read off the resulting figure, which is called the *standard drawing* of G .

A similar construction, without fixing the vertex a , gives rise to a class of graphs also called ‘permutation graphs’ (see [1]). In this paper, we only use this term as defined in Section 1.

The following lemma provides a link between induced paths in H_a and M -copies of P_{10} in G .

Lemma 4. *Suppose that H_a contains an induced path $xyzw$ on 4 vertices. Then $G\{a, x, y, z, w\}$ is an M - P_{10} in G .*

Proof. Let P be the path $xyzw$ in H_a . Since $xy \in E(P)$, the edges xx' and yy' cross. By symmetry, we may assume that $x \in aC'y$ and $y' \in a'C'x'$. First, note that $z \notin yCa$. Otherwise, as zz' crosses yy' , it would follow that zz' also crosses xx' , which contradicts the assumption that x and z are not adjacent in G . We now consider two cases, regarding whether or not $z \in aCx$.

Case 1: $z \in aCx$. Since the edges zz' and xx' do not cross, $z' \in a'C'x'$; moreover, since zz' and yy' cross, it follows that $z' \in y'C'x'$.

We assert that $w \in zCx$. Suppose that this is not the case. If $w \in aCz$, then ww' cannot cross zz' without crossing yy' , contradicting the fact that $zw \in E(P)$ and $yw \notin E(P)$. If $w \in xCy$, then ww' crosses xx' or yy' regardless of the position of w' , which results in a similar contradiction. Finally, if $w \in yCa$, then ww' cannot cross zz' without crossing xx' .

Thus, we have shown that $w \in zCx$, which implies that $w' \in a'C'y'$, as ww' and yy' do not cross. Summing up, $G\{a, x, y, z, w\}$ is precisely as in Figure 2 and constitutes an M -copy of the Petersen graph.

Case 2: $z \notin aCx$. Then, $z \in xCy$. Since zz' and xx' do not cross, $z' \in x'C'a'$. As ww' crosses zz' but none of xx' and yy' , the only possibility is that $w \in yCa$ and $w' \in x'C'z'$, which again produces an M - P_{10} . \square

By Lemma 4, if there is no M -copy of P_{10} containing aa' in G , then H_a contains no induced path on 4 vertices. Such graphs are known as *cographs* or *P_4 -free* graphs. There are various equivalent ways to describe them, summarized

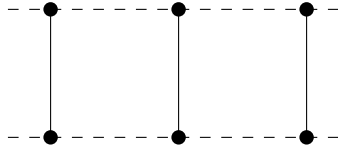


Figure 3: The triangular prism G with the unique 1-factor M such that (G, M) is a marked permutation graph.

in the survey [1, Theorem 11.3.3] by Brandstädt, Le and Spinrad. We use the characterisation that involves pairs of twin vertices. (Two vertices x and y of a graph H are *twins* if $N_H(x) = N_H(y)$.)

Theorem 5. *A graph G is P_4 -free if and only if every induced subgraph of G with at least two vertices contains a pair of twins.*

To be able to use Lemma 4 in conjunction with Theorem 5, we need to interpret twin pairs of H_a in terms of G .

Lemma 6. *Let x and y be twins in H_a . Let Q' be the path in G defined by*

$$Q' = \begin{cases} x'C'y' & \text{if } xy \notin E(H_a), \\ y'C'x' & \text{otherwise.} \end{cases}$$

Then M matches the vertices of the path xCy to those of Q' and vice versa.

Proof. Suppose, on the contrary, that the statement does not hold. By symmetry, we may assume that $x \in aCy$, and that M contains an edge $w w'$ with $w \in V(xCy)$ and $w' \notin V(Q')$. We assert that $w w'$ crosses exactly one edge from $\{xx', yy'\}$. To prove this, we consider two cases according to whether or not xx' and yy' cross. If they do not cross, then $w w'$ crosses only xx' (if $w' \in V(a'C'x')$) or only yy' (if $w' \in V(y'C'a')$). Otherwise, $w w'$ crosses only xx' (if $w' \in V(a'C'y')$) or only yy' (if $w' \in V(x'C'a')$). In each case, we obtain a contradiction with the assumption that x and y are twins in H_a . \square

We now prove Theorem 2, proceeding by induction on the number of vertices of G . The base case is the triangular prism, the unique permutation graph on 6 vertices (Figure 3), for which the theorem is trivially true since e cannot be contained in every M - C_4 of G . Therefore, we assume that G has at least 8 vertices and that every M -copy of C_4 in G contains the edge aa' .

Suppose first that $azz'a'$ is such an M -copy of C_4 . Let G_0 be the cubic graph obtained by removing the edge zz' and suppressing the resulting degree 2 vertices z and z' . Set $M_0 = M \setminus \{zz'\}$. All M_0 -copies of C_4 created by this operation contain the edge aa' . Therefore, regardless of whether or not G_0 contains an M_0 - C_4 , the induction hypothesis implies that aa' is contained in an M_0 -copy of P_{10} . This yields an M -copy of P_{10} in G containing aa' , as required.

Consequently, it may be assumed that G does not contain any M - C_4 . Assume that H_a contains no pair of twin vertices. Theorem 5 implies that H_a is not P_4 -free. Let X be a subset of $V(H_a)$ of size 4 such that $H_a[X] \simeq P_4$. By Lemma 4, $G\{X \cup \{a\}\}$ is an M -copy of P_{10} , and the sought conclusion follows.

Thus, we may assume that H_a contains twin vertices x and y . Without loss of generality, x belongs to aCy .

Let the path Q' be defined as in Lemma 6. Thus, since x and y are twins in H_a , vertices of the path xCy are only adjacent in M to vertices of Q' and *vice versa*. We transform G into another cubic graph G_1 by removing all vertices that are not contained in $\{a, a'\} \cup V(xCy \cup Q')$ and adding the edges $ax, ay, a'x'$ and $a'y'$ (if they are not present yet). Let M_1 be the perfect matching of G_1 consisting of all the edges of M contained in G_1 . Note that although the transformation may create M -copies of C_4 not present in G , the edge aa' is contained in every M_1 -copy of C_4 in G_1 . Furthermore, the path yCx in G must have some internal vertices other than a , since otherwise G would contain an M - C_4 , namely $yaa'y'$. Thus, G_1 has fewer vertices than G . The induction hypothesis implies that aa' is contained in an M_1 -copy of P_{10} in G_1 , and therefore also in G .

3 Counting the Petersen copies

Turning to the quantitative side of the question studied in Section 2, we now derive from Theorem 2 a lower bound on the number of M -copies of P_{10} in a permutation graph with no M - C_4 . The bound is linear in the order of the graph. We give a construction showing that this lower bound is tight up to a constant factor.

Throughout this section, (G, M) is a marked permutation graph with vertex set $A \cup A'$ just like in Section 2.

We will need two lemmas, the second of which we find to be of interest in its own right. The first lemma is an observation on auxiliary graphs which follows readily from the definition; its proof is omitted.

Lemma 7. *Let $a, b \in A$. Then the following hold for each $x, y \in A - \{a, b\}$:*

- (i) $ax \in H_b$ if and only if $bx \in H_a$,
- (ii) $xy \in H_b$ if and only if $|\{bx, by, xy\} \cap H_a| \in \{1, 3\}$.

Lemma 8. *Let $a, b \in A$. One of the following conditions holds:*

- *there is some M - P_{10} in G containing both aa' and bb' , or*
- *for any $F \subset A$ with $|F| = 4$ and $\{a, b\} \cap F = \emptyset$, it holds that $G\{F \cup \{a\}\} \simeq P_{10}$ if and only if $G\{F \cup \{b\}\} \simeq P_{10}$.*

Proof. Assume that there exists no M - P_{10} containing both aa' and bb' . By Lemma 4, it is sufficient to show that a set $\{u, w, x, y\} \subseteq A \setminus \{a, b\}$ induces a path of length 4 in H_a if and only if it induces a path of length 4 in H_b .

Let $U_1 = N_{H_a}(b) = N_{H_b}(a)$ and $U_2 = A \setminus (U_1 \cup \{a, b\})$. Lemma 4 implies that in the auxiliary graph H_a , there is no induced path of length 4 containing the vertex b . Therefore,

- (i) if $x, y \in U_1, z \in U_2$, and $xy \notin H_a$, then $xz \in H_a$ if and only if $yz \in H_a$, and
- (ii) if $x, y \in U_2, z \in U_1$, and $xy \in H_a$, then $xz \in H_a$ if and only if $yz \in H_a$.

Hence, if $uwxy$ is an induced path of length 4 in H_a , then $\{u, w, x, y\} \cap U_1 \in \{\{u, w, x, y\}, \{w, x\}, \emptyset\}$. By Lemma 7, it follows that $\{u, w, x, y\}$ induces a path of length 4 in H_b as well. More precisely, this path is $uwxy$ if $\{u, w, x, y\} \cap U_1 \in \{\{u, w, x, y\}, \emptyset\}$, and $uxwy$ if $\{u, w, x, y\} \cap U_1 = \{w, x\}$. The conclusion follows by symmetry of the roles played by a and b . \square

We can now prove the aforementioned lower bound.

Proposition 9. *If (G, M) is a marked permutation graph with $n \geq 40$ vertices and no M - C_4 , then (G, M) contains at least $n/2 - 4$ M -copies of the Petersen graph.*

Proof. If each edge of M is contained in at least 5 M -copies of P_{10} , the total number of copies is at least $(5n/2)/5 = n/2$. Hence, we may assume that there exists $x \in \{1, 2, 3, 4\}$ and an edge $e \in E(M)$ that is contained in only x M -copies of P_{10} . Let \mathcal{C} be the set of these copies.

At least $n/2 - 4x - 1$ edges of M are not contained in any M - P_{10} containing e . By Lemma 8, if we replace e by any such edge in any M - P_{10} from \mathcal{C} , we obtain an M - P_{10} again. These replacements yield $x(n/2 - 4x - 1)$ distinct M -copies of P_{10} . Thus, in total, (G, M) contains at least $x(n/2 - 4x)$ distinct M -copies of P_{10} . Minimizing this expression over $x \in \{1, 2, 3, 4\}$ and using the assumption that $n \geq 40$, we deduce that the number of copies is at least $n/2 - 4$, as asserted. \square

We now construct a family of marked permutation graphs (G_k, M_k) showing that the linear estimate in Proposition 9 is tight up to a constant factor. The graph G_k has $6k + 14$ vertices, contains no M_k - C_4 , and the number of M_k -copies of the Petersen graph in G_k is only $6k + 6$. (We note that graphs with a somewhat similar structure are constructed in [4, Section 3].)

For $k = 4$, the graph (G_k, M_k) is shown in Figure 4. We now give a formal definition and determine the number of M_k -copies of P_{10} .

Let $A = \{1, 2, \dots, 3k + 7\}$ and $A' = \{\bar{1}, \bar{2}, \dots, \overline{3k + 7}\}$. The vertex set of G_k is $A \cup A'$. On each of A and A' , we consider the standard linear order (in particular, $\bar{1} < \bar{2} < \dots < \overline{3k + 7}$). As in Section 2, we write i' for the neighbour in M_k of a vertex $i \in A$. Thus, $i' = \bar{j}$ for a suitable j .

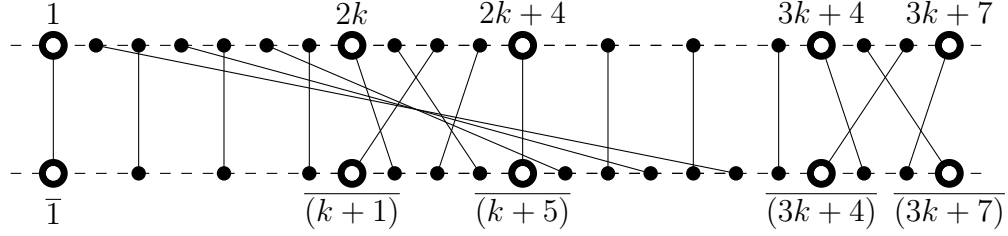


Figure 4: The marked permutation graph (G_4, M_4) . Labels are given only for the circled vertices.

Let

$$E_1 = \{(2i-1)\bar{i} : 1 \leq i \leq k\} \cup \{(2k+i+3)\overline{(k+2i+3)} : 1 \leq i \leq k\},$$

$$E_2 = \{(2i)\overline{(3k+4-2i)} : 1 \leq i \leq k-1\},$$

$$E_3 = \{(2k)\overline{(k+2)}, (2k+1)\overline{(k+4)}, (2k+2)\overline{(k+1)}, (2k+3)\overline{(k+3)}, \\ (3k+4)\overline{(3k+5)}, (3k+5)\overline{(3k+7)}, (3k+6)\overline{(3k+4)}, (3k+7)\overline{(3k+6)}\}.$$

Edges in E_1 , E_2 and E_3 will be called *vertical*, *skew* and *special*, respectively. Moreover, the first four and the last four edges in E_3 are two *groups of special edges*.

Proposition 10. *The marked permutation graph (G_k, M_k) contains exactly $6k+6$ M -copies of the Petersen graph.*

Proof. Each of the groups of special edges forms an M_k - P_{10} with each of the remaining $3k+3$ edges of M_k . We prove that besides these $6k+6$ copies, there are no other M_k -copies of P_{10} in G_k .

For $X \subseteq M_k$, we let G_X be the graph obtained from $G_k - M_k$ by adding the edges in X and suppressing the degree 2 vertices.

Let X be a subset of M_k that contains no group of special edges. Suppose that $G_k\{X\}$ is isomorphic to the Petersen graph. To obtain a contradiction, we show that $G_k\{X\}$ contains a 4-cycle.

First of all, if X contains a special edge, then it contains no other special edge from the same group. Indeed, a quick case analysis shows that if Y consists of any two or three special edges in the same group, then G_Y contains a Y - C_4 .

Thus, for the purposes of our argument, special edges behave just like vertical ones. We assert next that X contains at most one skew edge. Let $j_1j'_1$ and $j_2j'_2$ be skew edges with $j_1 < j_2$ and $j_1 + j_2$ maximum among the skew edges in X .

Observe that X contains no vertical edge ii' with $i > j_2$ and $i' < j'_2$. Indeed, if there is only one such edge, then it forms an $X-C_4$ in G_X together with $j_2j'_2$, while if there are at least two such edges, then an $X-C_4$ is obtained from a consecutive pair among them.

By a similar argument, X contains neither any vertical edge ii' with $j_1 < i < j_2$, nor any vertical edge ii' with $j'_2 < i' < j'_1$. It follows that $j_1j'_1$ and $j_2j'_2$ are contained in an $X-C_4$ in G_X , a contradiction which proves that there is at most one skew edge in X .

Consequently, X contains a set Y of at least four edges that are vertical or special, as $|X| = 5$. Further, $|Y| \neq 5$, so there are exactly four Y -copies of C_4 in G_Y . Only at most two of these will be affected by the addition of the fifth edge of X . Thus, an $X-C_4$ persists in G_X , a contradiction. The proof is complete. \square

While the graphs constructed in the proof of Proposition 10 are C_4 -free, they are not cyclically 5-edge connected. A slight modification of the construction ensures this stronger property, but makes the discussion somewhat more complicated. For this reason, we only described the simpler version.

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