Antimagic Properties of Graphs with Large Maximum Degree

Zelealem B. Yilma
Department of Mathematical Sciences
Carnegie Mellon University
Pittsburgh, PA 15213
E-mail: zyilma@andrew.cmu.edu

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Abstract

An *antimagic labeling* of a graph with m edges and n vertices is a bijection from the set of edges to the integers $1, \ldots, m$ such that all n vertex sums are pairwise distinct, where a vertex sum is the sum of labels of all edges incident with the same vertex. A graph is called *antimagic* if it has an antimagic labeling. In this paper we discuss antimagic properties of graphs which contain vertices of large degree. We also show that graphs with maximum degree at least $n-3$ are antimagic.

1 Introduction

All graphs in this paper are finite, undirected and simple. To avoid repetition, unless specified otherwise, a graph G has m edges and n vertices. We denote by $\Gamma_G(v)$ and $d_G(v)$ (dropping the subscripts when the graph G is clear from context) the neighborhood and the degree, respectively, of a vertex $v \in V(G)$.

For some $S \subseteq \mathbb{Z}$, let $\tau : E(G) \to S$ be a labeling of the edges of a graph G. The labeling τ induces a weight, $w_\tau : V(G) \to \mathbb{Z}$, on the vertices of G, where $w_\tau(v) = \sum_{uv \in E} \tau(uv)$. One may then specify the set S and/or put restrictions on the function τ and ask if w_τ satisfies a specified property. There are various results and conjectures using this setup (see, for example, [6, 7]). In this paper, we consider labelings where $S = \{1, \ldots, m\}$ and τ is a bijection. Such a labeling is called *antimagic* if $w_\tau(v) \neq w_\tau(u)$ for all distinct $u, v \in V$. The graph G is *antimagic* if it permits an antimagic labeling.

It is conjectured in [3] that

Conjecture 1.1. *Every connected graph, but K_2, is antimagic.*

While the general question is still open, probabilistic (Alon, Kaplan, Roditty, and Yuster [1]), combinatorial (Cranston [2]), and algebraic (Hefetz [4]; Hefetz, Saluz, and Tran [5]) arguments have been used to confirm the conjecture for certain classes of graphs. One such class is that of graphs with large maximum degree. In [1], Alon, et al., show the following result.

Theorem 1.2. *If G has $n > 4$ vertices and $\Delta(G) \geq n - 2$ then G is antimagic.*

They also note that it is still open whether every connected graph with $\Delta(G) \geq n-k$ and $n > n_0(k)$ is antimagic.

In this paper we provide a simple constructive proof of the following:

Theorem 1.3. *If G is a graph on n vertices, $\Delta(G) = d(x) = n-k$ where $k \leq n/3$, and there exists $y \in V$ such that $\Gamma(x) \cup \Gamma(y) = V$, then G is antimagic.*

This not only provides a simpler proof of Theorem 1.2 but arguments in the proof may be modified to show

Theorem 1.4. *If G is connected, has $n \geq 9$ vertices, and $\Delta(G) \geq n-3$, then G is antimagic.*

In the next section, we present a lemma which provides us with the basic framework that we exploit in proving both Theorem 1.3 (Section 3) and Theorem 1.4 (Section 4).

2 Framework

We reproduce the proof of this well known result which also appears in [1].

Lemma 2.1. *If $\Delta(G) = n-1$, then G is antimagic.*

Proof. Let v_0 be a vertex of degree $n-1$ and let T be a breadth-first spanning tree rooted at v_0 (in this case, T is a star). Let $G' = G \setminus T$. Let $\tau' : E(G') \to \{1, \ldots, m-n+1\}$ be an arbitrary (bijective) labeling of the edges of G' and let $w' = w_{\tau'}$. Order the $n-1$ vertices $V \setminus \{v_0\}$ such that $w'(v_i) \geq w'(v_j)$ for $1 \leq i < j \leq n-1$. Let $\tau : E \to \{1, \ldots, m\}$ be an extension of τ' such that $\tau(v_0 v_i) = m - i + 1$ for $i = 1, \ldots, n-1$. Then, for $i < j$,

$$w_\tau(v_i) = w'(v_i) + m + 1 - i \geq w'(v_j) + m + 1 - i > w'(v_j) + m + 1 - j = w_\tau(v_j).$$

In addition, edges incident to v_0 receive the largest $n-1$ labels. Thus, $w_\tau(v_i) > w_\tau(v_j)$ for all $0 \leq i < j \leq n-1$ and G is antimagic. □

The above proof highlights the major steps in proving Theorems 1.3 and 1.4. The key idea is to isolate a breadth-first spanning tree T and reserve the largest $n-1$ labels for it. First of all, this guarantees (given lower bounds on $\Delta(G)$) that the root vertex has the highest possible vertex sum. In addition, after arbitrarily labeling the remaining edges (or, in general, assigning arbitrary, but bounded, partial weights to the other vertices), we may label the edges of T so that all other vertex sums are distinct.

However, when $\Delta(G) < n-1$, the above simple labeling of T may not suffice. To fix this problem we alter the above labeling by *shifting* the labels on some edges. We prove that this shifting procedure remedies some instances of the problem when T has at most two non-root vertices which are not leaves. The analysis seems to get messy after that.

3 Proof of Theorem 1.3

Let x be a vertex of degree $\Delta(G) = n-k$, where $k \leq n/3$, and let y be such that $\Gamma(x) \cup \Gamma(y) = V$. Let T be a breadth-first spanning tree rooted at x and visiting y first. Therefore, all vertices in $V \setminus \{x, y\}$ are leaves in T and the degree sequence of T is $(n-k, k, 1, \ldots, 1)$. Let $G' = G \setminus T$. Fix an arbitrary bijection $\tau' : E(G') \to \{1, \ldots, m-n+1\}$ and let $w' = w_{\tau'}$.

Lemma 3.1. *For any extension $\tau : E(G) \to \{1, \ldots, m\}$ of τ' and all $u \neq x$, $w_\tau(x) > w_\tau(u)$.*

Proof of Lemma. First note that

$$w_\tau(x) \geq \sum_{i=1}^{n-k} (m - n + 1 + i).$$

Let $u \in V$, $u \neq x$. Then,

$$\begin{aligned}
w_\tau(u) &= w'(u) + \sum_{v \in \Gamma_T(u)} \tau(uv) \\
&\leq \sum_{i=0}^{d_{G'}(u)-1} (m - n + 1 - i) + \sum_{i=0}^{d_T(u)-1} (m - i) && (1) \\
&\leq \sum_{i=0}^{n-2k-1} (m - n + 1 - i) + \sum_{i=0}^{k-1} (m - i) && (2) \\
&= \sum_{i=0}^{n-2k-1} (m - n + 2 + i) + \sum_{i=n-2k}^{n-k-1} (m - n + 2 + i) - (n - 2k)^2 + 2\binom{k}{2} \\
&\leq \sum_{i=0}^{n-k-1} (m - n + 2 + i) - k^2 + 2\binom{k}{2} && (3) \\
&\leq w_\tau(x) - k.
\end{aligned}$$

In (1), we bound $w_\tau(u)$ from above by taking the largest $d_{G'}(u)$ labels from $\{1, \ldots, m - n + 1\}$ and the largest $d_T(u)$ labels from $\{m - n + 2, \ldots, m\}$. However, as $d_G(u) = d_{G'}(u) + d_T(u) \leq n - k$, this sum is maximized when $d_T(u)$ is as large as possible. We then obtain (2) by noting that $d_T(u) \leq k$ for $u \neq x$. We rewrite the expression in a more convenient form and use the fact that $n \geq 3k$ to obtain (3), thereby proving the lemma. □

Now let $V' = V \setminus \{x, y\} = \{v_1, \ldots, v_{n-2}\}$ where the vertices $v_j \in V'$ are ordered such that $w'(v_j) \geq w'(v_k)$ for $1 \leq j < k \leq n - 2$. Let e_j denote the unique, as yet unlabeled, edge incident to v_j in T (so, $e_j = xv_j$ or $e_j = yv_j$). As in the proof of Lemma 2.1, we want an extension τ for which $\tau(e_j) > \tau(e_k)$ whenever $j < k$. This gives us one degree of freedom for the label of edge xy. To this end, we define a sequence of extensions $\tau_i : E \to [m]$ of τ' for $i = 0, \ldots, n - 2$ where

$$\tau_i(e) = \begin{cases} m - i, & \text{if } e = xy \\ m - j + 1 & \text{if } e = e_j,\ j \leq i \\ m - j, & \text{if } e = e_j,\ j > i. \end{cases}$$

Notice that τ_i may be obtained from τ_{i-1} by exchanging the labels on xy and e_i, essentially incrementing the vertex sum at v_i and potentially decrementing the vertex sum at y. Letting $w_i = w_{\tau_i}$, we immediately observe that $w_i(v_j) > w_i(v_k)$ for all i whenever $j < k$. Therefore, we need only to show that the vertex sum at y is distinct from vertex sums of vertices in V' for some extension τ_q. We not only show the existence of such an index q, but we find one in which y is, in some sense, in its "natural position". To be precise, we pick q such that the order imposed on the edges $\{xy, e_1, \ldots, e_{n-2}\}$ by τ_q matches the order imposed on the vertices $\{y, v_1, \ldots, v_{n-2}\}$ by w_q. In other words, as the label $\tau_q(xy) = m - q < \tau_q(e_i)$ for q edges $e_i \in T$, we also have exactly q vertices $v_i \in V'$ whose vertex sum exceeds that of y.

Let $\mathcal{I} = \{i \ : \ w_i(y) > w_i(v_{i+1})\}$ and let $\mathcal{I}^* = \mathcal{I} \cup \{n-2\}$. Let q be the smallest member of $\mathcal{I}^*$. If $q = 0$, then $w_0(y) > w_0(v_1) \ldots > w_0(v_{n-2})$. Otherwise,

$$w_q(v_{q+1}) < w_q(y) \leq w_{q-1}(y) \leq w_{q-1}(v_q) = w_q(v_q) - 1 < w_q(v_q).$$

Therefore, τ_q is an antimagic labeling and G is antimagic, completing the proof of Theorem 1.3.

Corollary 3.2. *If $\Delta(G) = n-2$, then G is antimagic.*

Proof. If G is connected, the proof follows from Theorem 1.3. Otherwise, G has an isolated vertex and a component G'' on $n'' = n-1$ vertices with $\Delta(G'') = n'' - 1$. Then the result follows from Lemma 2.1. □

4 Proof of Theorem 1.4

Let x be a vertex of degree $n-3$ and let a, b be the two non-neighbors of x. We divide the proof into 3 cases, two of which are easily resolved by applying (a slight modification of) the argument in Theorem 1.3. The proof of the third case follows in a similar vein but is a bit more involved. The restriction that $n \geq 9$ is needed here as we apply a variant of Lemma 3.1 (with $k = 3$). It is simple (but time consuming) to verify the theorem for $n < 9$.

Case 1: There is a vertex y such that $a, b \in \Gamma(y)$.

This follows via a direct application of Theorem 1.3.

Case 2: $\Gamma(a) = \{b\}$.

Assign the label 1 to the edge ab. As the vertex sum at a will be 1, it is guaranteed to be less than that of other vertices. A slight modification of the argument from Theorem 1.3 can then be applied to $G' = G \setminus \{a\}$.

Case 3: Cases 1 and 2 do not hold.

As G is connected, there are two vertices $y_1 \neq b$, $y_2 \neq a$ such that $ay_1, by_2 \in E(G)$. Let T be a breadth-first spanning tree rooted at x and visiting y_1 and y_2 first. Observe that the degree sequence in T is $(n-3, 2, 2, 1, \ldots, 1)$. Let $G' = G \backslash T$, fix an arbitrary bijection $\tau' : E(G') \to \{1, \ldots, m-n+1\}$, and retain the largest $n-1$ labels for the edges in T. Let $w' = w_{\tau'}$ be the induced partial weight after this initial labeling. Even though T has three non-leaves, the argument in Lemma 3.1 applies and $w_\tau(x) > w_\tau(u)$ for all $u \in V, u \neq x$ and all extensions $\tau : E(G) \to \{1, \ldots, m\}$ of τ'.

Let $V' = V \setminus \{x, y_1, y_2\} = \{v_1, \ldots, v_{n-3}\}$ where the vertices $v_j \in V'$ are ordered such that $w'(v_j) \geq w'(v_{j+1})$. Let e_j be the unique edge in T incident to v_j. To obtain an antimagic labeling, we once again look for an extension τ satisfying $\tau(e_j) > \tau(e_k)$ whenever $j < k$. However, this time, we have two degrees of freedom as we are free to choose the labels on xy_1 and xy_2. We begin by defining the following $(n-1)(n-2)$ different extensions of τ', and apply an argument like the one used in the proof of Theorem 1.3, although modifications are required in some cases.

$$\text{Let } \tau_{i,j} : E \to \{1, \ldots, m\}, \text{ where } i, j \in \{0, \ldots, n-2\}, i \neq j \text{ satisfy}$$

$$\tau_{i,j}(e) = \begin{cases} \tau'(e) & \text{if } e \in G' \\ m-i, & \text{if } e = xy_1 \\ m-j, & \text{if } e = xy_2 \\ m-t+1, & \text{if } e = e_t;\ t \leq \min(i,j) \\ m-t, & \text{if } e = e_t;\ \min(i,j) < t < \max(i,j) \text{ and} \\ m-t-1, & \text{if } e = e_t;\ t \geq \max(i,j) \end{cases}$$

Let $w_j^i = w_{\tau_{i,j}}$. We first observe that $w_j^i(v_k) > w_j^i(v_l)$ whenever $k < l$. Therefore, we need only find a pair p, q for which $w_q^p(y_1)$ and $w_q^p(y_2)$ are unequal and distinct from $w_q^p(v_k)$ for all $k \in \{1, \ldots, n-3\}$. We find an index q by starting from $\tau_{0,1}$ and considering extensions of the form $\tau_{0,j}$, that is, by shifting the label on xy_2 to find a "natural position" for y_2. Next, we range over the first index to find a suitable value for p.

Let $\mathcal{J} = \{j \ : \ w_j^0(y_2) > w_j^0(v_j)\}$ and $\mathcal{J}^* = \mathcal{J} \cup \{n-2\}$. Let q be the smallest member of $\mathcal{J}^*$. Now let $U = V' \cup \{y_2\} = \{u_1, \ldots, u_{n-2}\}$ where

$$u_k = \begin{cases} v_k, & \text{for } k < q \\ y_2, & \text{for } k = q \\ v_{k-1}, & \text{for } k > q. \end{cases} \quad \text{and}$$

Observe that, after renaming the vertices, we have $w_q^0(u_k) > w_q^0(u_{k+1})$ for all $1 \leq k \leq n-3$. Furthermore, under $\tau_{0,q}$, the edge joining a vertex u_k to its parent in T receives the label $m-k$. Now, with $\tau_{0,q}$ as our new starting point, we shift the label on xy_1 by considering extensions of the form $\tau_{i,q'}$ where $q' = q$ for $i < q$ and $q' = q-1$ for $i \geq q$. Let $\mathcal{I} = \{i : w_{q'}^i(y_1) > w_{q'}^i(u_{i+1})\}$ and $\mathcal{I}^* = \mathcal{I} \cup \{n-2\}$. Let p be the smallest member of $\mathcal{I}^*$. The choice of p guarantees that

$$w_{q'}^p(u_p) > w_{q'}^p(y_1) > w_{q'}^p(u_{p+1}).$$

However, note that as $u_q = y_2$ is not a leaf in T, its weight is incremented not only when $i = q$ (that is, the label on edge xy_2 is incremented) but also when $u_i = b$ (the label on edge by_2 is incremented). It is possible, therefore, that $w_{q'}^p(u_q) \geq w_{q'}^p(u_{q-1})$. We check for and correct these instances by consider the following cases: (for brevity's sake, we have chosen not to segregate the cases where $q' = 0$. Inequalities seemingly referring to u_{-1} are to be considered void.)

Case A: $u_{q-1} \neq b$.

Here, $\tau_{p,q'}$ itself is an antimagic labeling. Note that, by definition of $\mathcal{J}$, we have $w_{q-1}^0(u_{q-1}) \geq w_{q-1}^0(y_2)$, and, as $u_{q-1} \neq b$, it follows that $w_q^0(u_{q-1}) \geq w_q^0(y_2) + 2$ (the shift increments the label on e_{q-1} and decrements that of xy_2). Hence, if $p < q$, we take $\tau_{0,q}$ as a reference point and obtain

$$w_q^p(u_{q-1}) \geq w_q^0(u_{q-1}) \geq w_q^0(y_2) + 2 \geq w_q^p(y_2) + 1.$$

On the other hand, if $p \geq q$, taking $\tau_{0,q-1}$ as our starting point, we see

$$w_{q-1}^p(u_{q-1}) = w_{q-1}^0(u_{q-1}) + 2 \geq w_{q-1}^0(y_2) + 2 \geq w_{q-1}^p(y_2) + 1.$$

Case B: $u_{q-1} = b$

We use $\tau_{0,q}$ as our reference point. Note that, in this case, we can only guarantee

$$w_q^0(b) \geq w_q^0(y_2) + 1.$$

Further note that the label on by_2 is incremented if $p \geq q-1$ and the label on xy_2 is incremented if $p \geq q$.

So, if $p < q-1$, the label on neither of the edges by_2 and xy_2 is affected and $w_q^p(b) \geq w_q^p(y_2)+1$.

If $p = q-1$, both vertex sums are incremented by one and the required inequality still holds.

The remaining case is then $p \geq q$. Note here that the labels on both by_2 and xy_2 have been incremented and, thus, the vertex sum at b goes up by 1 whereas that of y_2 goes up by 2, potentially causing an overlap. If so, that is, if $w_{q-1}^p(b) = w_{q-1}^p(y_2)$, consider instead the labeling $\tau_{p,q-2}$, essentially swapping the labels on by_2 and xy_2. This swap decreases the vertex sum at b by 1 and leaves all other vertex sums unchanged, thereby avoiding the conflict (see Figure 1).

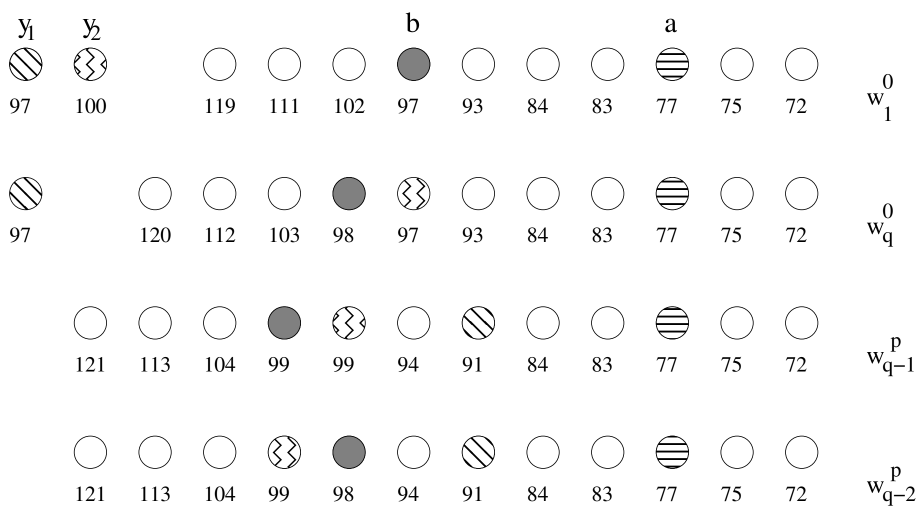

Figure 1: Illustrative example for shifting. Note that $w_{q-1}^p(b) = w_{q-1}^p(y_2)$.

This completes the proof of Theorem 1.4.

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