21-241: Matrix Algebra – Summer I, 2006

Quiz 5 Solutions

1. (15 points) Let $A = \begin{pmatrix} 4 & 0 & 1 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{pmatrix}$.

   (a) Find the characteristic polynomial of A.
   SOLUTION.

$$\begin{aligned} \det(A - \lambda I) &= \det \begin{pmatrix} 4-\lambda & 0 & 1 \\ -2 & 1-\lambda & 0 \\ -2 & 0 & 1-\lambda \end{pmatrix} \\ &= (1-\lambda) \det \begin{pmatrix} 4-\lambda & 1 \\ -2 & 1-\lambda \end{pmatrix} \quad \text{(cofactor expansion across the second column)} \\ &= (1-\lambda)[(4-\lambda)(1-\lambda)+2] \\ &= -\lambda^3 + 6\lambda^2 - 11\lambda + 6. \end{aligned}$$

   (b) It's known that 1 is an eigenvalue of A. Find a basis for the eigenspace corresponding to 1.
   SOLUTION. The eigenspace corresponding to 1 is the kernel of $A - I$.

$$A - I = \begin{pmatrix} 3 & 0 & 1 \\ -2 & 0 & 0 \\ -2 & 0 & 0 \end{pmatrix} \xrightarrow[R_3+\frac{2}{3}R_1]{R_2+\frac{2}{3}R_1} \begin{pmatrix} 3 & 0 & 1 \\ 0 & 0 & \frac{2}{3} \\ 0 & 0 & \frac{2}{3} \end{pmatrix} \xrightarrow{R_3-R_2} \begin{pmatrix} \boxed{3} & 0 & 1 \\ 0 & 0 & \boxed{\frac{2}{3}} \\ 0 & 0 & 0 \end{pmatrix}$$

   Therefore $x_3 = 0$, $x_1 = -\frac{1}{3}x_3 = 0$, x_2 is free. The general element in $\ker(A-I)$ is

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ x_2 \\ 0 \end{pmatrix} = x_2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}.$$

   Thus $(0, 1, 0)^T$ is a basis for the eigenspace corresponding to 1.

   (c) Find all eigenvalues of A. What are their multiplicities?
   SOLUTION. Since 1 is an eigenvalue, $(\lambda - 1)$ is a factor of the characteristic polynomial. Thus,

$$\det(A - \lambda I) = -\lambda^3 + 6\lambda^2 - 11\lambda + 6 = -(\lambda-1)(\lambda^2 - 5\lambda + 6) = -(\lambda-1)(\lambda-2)(\lambda-3).$$

   Therefore, A has three simple eigenvalues 1, 2 and 3. □

2. (5 points) Let λ be an eigenvalue of an invertible matrix A. Show that λ^{-1} is an eigenvalue of A^{-1}.
   PROOF. Since λ is an eigenvalue of A, there exists a nonzero vector $\mathbf{v}$ such that

$$A\mathbf{v} = \lambda\mathbf{v}.$$

   Since A is invertible, A^{-1} exists and $\lambda \neq 0$. Multiplying both sides of the above equation by $\lambda^{-1}A^{-1}$, we get

$$\lambda^{-1}\mathbf{v} = A^{-1}\mathbf{v}.$$

   This shows that λ^{-1} is an eigenvalue of A^{-1} and $\mathbf{v}$ is an eigenvector corresponding to λ^{-1}. □