21-241: Matrix Algebra – Summer I, 2006

Quiz 4 Solutions

1. (15 points) Let $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$, $\mathbf{v}_2 = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$, $\mathbf{v}_3 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$. Verify that they form a basis for $\mathbb{R}^3$. Starting with $\mathbf{v}_1,\ \mathbf{v}_2,\ \mathbf{v}_3$ and using the Euclidean dot product, construct an orthonormal basis on $\mathbb{R}^3$.

   Solution. Combine $\mathbf{v}_1,\ \mathbf{v}_2,\ \mathbf{v}_3$ into a matrix and reduce it in the echelon form

$$\begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ -1 & 2 & 0 \end{pmatrix} \xrightarrow[R_3+R_1]{R_2-R_1} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{pmatrix} \xrightarrow{R_3-2R_2} \begin{pmatrix} \boxed{1} & 0 & 1 \\ 0 & \boxed{1} & 0 \\ 0 & 0 & \boxed{1} \end{pmatrix}$$

   We see that each column contains a pivot, so $\mathbf{v}_1,\ \mathbf{v}_2,\ \mathbf{v}_3$ are linearly independent. Thus they form a basis for $\mathbb{R}^3$. By Gram-Schmidt formula, we construct an orthogonal basis $\mathbf{w}_1,\ \mathbf{w}_2,\ \mathbf{w}_3$ as follows:

$$\mathbf{w}_1 = \mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix},$$

$$\mathbf{w}_2 = \mathbf{v}_2 - \frac{\mathbf{v}_2 \cdot \mathbf{w}_1}{\|\mathbf{w}_1\|^2}\,\mathbf{w}_1 = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} - \frac{-1}{3}\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1/3 \\ 4/3 \\ 5/3 \end{pmatrix},$$

$$\mathbf{w}_3 = \mathbf{v}_3 - \frac{\mathbf{v}_3 \cdot \mathbf{w}_1}{\|\mathbf{w}_1\|^2}\,\mathbf{w}_1 - \frac{\mathbf{v}_3 \cdot \mathbf{w}_2}{\|\mathbf{w}_2\|^2}\,\mathbf{w}_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} - \frac{2}{3}\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} - \frac{5/3}{42/9}\begin{pmatrix} 1/3 \\ 4/3 \\ 5/3 \end{pmatrix} = \begin{pmatrix} 3/14 \\ -1/7 \\ 1/14 \end{pmatrix}.$$

   We finally normalize $\mathbf{w}_1,\ \mathbf{w}_2,\ \mathbf{w}_3$ to obtain an orthonormal basis:

$$\mathbf{u}_1 = \frac{\mathbf{w}_1}{\|\mathbf{w}_1\|} = \begin{pmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ -1/\sqrt{3} \end{pmatrix}, \qquad \mathbf{u}_2 = \frac{\mathbf{w}_2}{\|\mathbf{w}_2\|} = \begin{pmatrix} 1/\sqrt{42} \\ 4/\sqrt{42} \\ 5/\sqrt{42} \end{pmatrix}, \qquad \mathbf{u}_3 = \frac{\mathbf{w}_3}{\|\mathbf{w}_3\|} = \begin{pmatrix} 3/\sqrt{14} \\ -2/\sqrt{14} \\ 1/\sqrt{14} \end{pmatrix}.$$

   □

2. (5 points) Instead of using the Euclidean dot product on $\mathbb{R}^3$, we now use the weighted inner product $\langle \mathbf{u}, \mathbf{v} \rangle = u_1v_1 + u_2v_2 + 2u_3v_3$. Starting with $\mathbf{v}_1,\ \mathbf{v}_2,\ \mathbf{v}_3$, construct an orthogonal basis on $\mathbb{R}^3$.

   Solution. Replacing the dot product by the weighted inner product in the Gram-Schmidt formula, we recompute $\mathbf{w}_1,\ \mathbf{w}_2,\ \mathbf{w}_3$ as follows:

$$\mathbf{w}_1 = \mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix},$$

$$\mathbf{w}_2 = \mathbf{v}_2 - \frac{\langle \mathbf{v}_2, \mathbf{w}_1 \rangle}{\|\mathbf{w}_1\|^2}\,\mathbf{w}_1 = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} - \frac{-3}{4}\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 3/4 \\ 7/4 \\ 5/4 \end{pmatrix},$$

$$\mathbf{w}_3 = \mathbf{v}_3 - \frac{\langle \mathbf{v}_3, \mathbf{w}_1 \rangle}{\|\mathbf{w}_1\|^2}\,\mathbf{w}_1 - \frac{\langle \mathbf{v}_3, \mathbf{w}_2 \rangle}{\|\mathbf{w}_2\|^2}\,\mathbf{w}_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} - \frac{2}{4}\begin{pmatrix} 3/4 \\ 7/4 \\ 5/4 \end{pmatrix} - \frac{5/2}{27/4}\begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 2/9 \\ -4/27 \\ 1/27 \end{pmatrix}.$$

   □