21-241: Matrix Algebra – Summer I, 2006

Quiz 3 Solutions

1. (10 points) Classify the quadratic form $q(\mathbf{x}) = x_1^2 + 3x_2^2 + 4x_3^2 + x_4^2 + 2x_1x_2 - 4x_1x_4 + 8x_2x_3 + 9x_3x_4$.

   Solution. The associated matrix $K = \begin{pmatrix} 1 & 1 & 0 & -2 \\ 1 & 3 & 4 & 0 \\ 0 & 4 & 4 & \frac{9}{2} \\ -2 & 0 & \frac{9}{2} & 1 \end{pmatrix}$. Applying Gaussian to K, we see that

$$\begin{pmatrix} 1 & 1 & 0 & -2 \\ 1 & 3 & 4 & 0 \\ 0 & 4 & 4 & \frac{9}{2} \\ -2 & 0 & \frac{9}{2} & 1 \end{pmatrix} \xrightarrow[R_4+2R_1]{R_2-R_1} \begin{pmatrix} 1 & 1 & 0 & -2 \\ 0 & 2 & 4 & 2 \\ 0 & 4 & 4 & \frac{9}{2} \\ 0 & 2 & \frac{9}{2} & -3 \end{pmatrix} \xrightarrow[R_4-R_2]{R_3-2R_2} \begin{pmatrix} \boxed{1} & 1 & 0 & -2 \\ 0 & \boxed{2} & 4 & 2 \\ 0 & 0 & \boxed{-4} & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} & -5 \end{pmatrix}.$$

   Hence, K has both positive and negative pivots. Thus the quadratic form q is indefinite. $\square$

2. (10 points) Let $q(x, y, z) = x^2 + y^2 + yz + z^2 + x + y - z + 4$. Determine if q has a minimum. If so, find the minimizer and the minimum value for q.

   Solution. The quadratic form $q(x, y, z)$ can be written as

$$q(\mathbf{u}) = \mathbf{u}^T K \mathbf{u} - 2\mathbf{u}^T \mathbf{f} + c,$$

   where $\mathbf{u} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$, $K = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & \frac{1}{2} \\ 0 & \frac{1}{2} & 1 \end{pmatrix}$, $\mathbf{f} = \begin{pmatrix} -\frac{1}{2} \\ -\frac{1}{2} \\ \frac{1}{2} \end{pmatrix}$, $c = 4$. Applying Gaussian to the augmented matrix $(K|\mathbf{f})$, we get

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & -\frac{1}{2} \\ 0 & 1 & \frac{1}{2} & -\frac{1}{2} \\ 0 & \frac{1}{2} & 1 & \frac{1}{2} \end{array}\right) \xrightarrow{R_3-\frac{1}{2}R_2} \left(\begin{array}{ccc|c} \boxed{1} & 0 & 0 & -\frac{1}{2} \\ 0 & \boxed{1} & \frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & \boxed{\frac{3}{4}} & \frac{3}{4} \end{array}\right),$$

   all three pivots are positive. Therefore, q has a minimum, the minimizer $\mathbf{u}^*$ is the unique solution to the system $K\mathbf{u} = \mathbf{f}$. By back substitution, we obtain $\mathbf{u}^* = (-\frac{1}{2}, -1, 1)^T$. Thus the minimum value is $q(\mathbf{u}^*) = c - \mathbf{u}^{*T}\mathbf{f} = \frac{11}{4}$. $\square$