21-241: Matrix Algebra – Summer I, 2006

Quiz 2

1. (10 points) Let W be the set of all vectors of the form $\begin{pmatrix} s+3t \\ s-t \\ 2s-t \\ 4t \end{pmatrix}$, where s and t are arbitrary. Show that W is a subspace of $\mathbb{R}^4$.

   Proof. Clearly W is nonempty. Then you may directly prove W is closed under addition and scalar multiplication. But the following method is more convenient. Since

$$\begin{pmatrix} s+3t \\ s-t \\ 2s-t \\ 4t \end{pmatrix} = \begin{pmatrix} s \\ s \\ 2s \\ 0 \end{pmatrix} + \begin{pmatrix} 3t \\ -t \\ -t \\ 4t \end{pmatrix} = s\begin{pmatrix} 1 \\ 1 \\ 2 \\ 0 \end{pmatrix} + t\begin{pmatrix} 3 \\ -1 \\ -1 \\ 4 \end{pmatrix} = s\mathbf{u} + t\mathbf{v},$$

   where $\mathbf{u} = (1, 1, 2, 0)^T$, $\mathbf{v} = (3, -1, -1, 4)^T$, we know that $W = \operatorname{span}\{\mathbf{u}, \mathbf{v}\}$, which has to be a subspace of $\mathbb{R}^4$. □

2. (10 points) Find a basis for the set of vectors in $\mathbb{R}^3$ in the plane $x + 2y + z = 0$.

   Solution. Any vector in this plane is actually a solution to the homogeneous system $x + 2y + z = 0$ (although this system contains only one equation). So we are to find a basis for the kernel of the coefficient matrix $A = \begin{pmatrix} 1 & 2 & 1 \end{pmatrix}$, which is already in the echelon form. Clearly, y and z are free variables, and $x = -2y - z$. So the general solution can be written as

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -2y-z \\ y \\ z \end{pmatrix} = \begin{pmatrix} -2y \\ y \\ 0 \end{pmatrix} + \begin{pmatrix} -z \\ 0 \\ z \end{pmatrix} = y\begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} + z\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}.$$

   Therefore, $\{(-2, 1, 0)^T, (-1, 0, 1)^T\}$ form a basis of $\ker A$, namely the set of vectors in the plane $x + 2y + z = 0$. □