21-241: Matrix Algebra – Summer I, 2006

Quiz 1 Solutions

1. (10 points) The *trace* of $n \times n$ matrix A is defined to be the sum of its diagonal entries: $\text{tr}A = a_{11} + a_{22} + \cdots + a_{nn}$. Suppose matrix P has size $m \times n$ and matrix Q has size $n \times m$. Prove that $\text{tr}(PQ) = \text{tr}(QP)$. (If you are not comfortable to deal with symbols m and n, prove the statement for $m = 2$, $n = 3$, at the price of 1 point off.)

   SOLUTION. Matrix PQ has size $m \times m$, matrix QP has size $n \times n$. Both of them are square, so their traces are well defined. Let $A = PQ$, $B = QP$. We only care about the diagonal entries of A and B. By the rule of matrix multiplication, a_{ii}, the i-th diagonal entry of A, is the vector product of the i-th row of P and the i-th column of Q, so

$$a_{ii} = \begin{pmatrix} p_{i1} & p_{i2} & \cdots & p_{in} \end{pmatrix} \begin{pmatrix} q_{1i} \\ q_{2i} \\ \vdots \\ q_{ni} \end{pmatrix} = \sum_{j=1}^{n} p_{ij} q_{ji} \qquad \forall\, i = 1, 2, \cdots, m.$$

   Therefore, by definition of trace,

$$\text{tr}(A) = \sum_{i=1}^{m} a_{ii} = \sum_{i=1}^{m} \left(\sum_{j=1}^{n} p_{ij} q_{ji} \right) = \sum_{\substack{i=1,2,\cdots,m \\ j=1,2,\cdots,n}} p_{ij} q_{ji}.$$

   A similar argument tells us that b_{jj}, the j-th diagonal entry of B, equals

$$b_{jj} = \begin{pmatrix} q_{j1} & q_{j2} & \cdots & q_{jm} \end{pmatrix} \begin{pmatrix} p_{1j} \\ p_{2j} \\ \vdots \\ p_{mj} \end{pmatrix} = \sum_{i=1}^{m} q_{ji} p_{ij} \qquad \forall\, j = 1, 2, \cdots, n,$$

   and so,

$$\text{tr}(B) = \sum_{j=1}^{n} b_{jj} = \sum_{j=1}^{n} \left(\sum_{i=1}^{m} q_{ji} p_{ij} \right) = \sum_{\substack{i=1,2,\cdots,m \\ j=1,2,\cdots,n}} q_{ji} p_{ij}.$$

   Now, apparently, $\text{tr}(A) = \text{tr}(B)$, i.e., $\text{tr}(PQ) = \text{tr}(QP)$. □

2. (10 points) Find two different permuted LU factorizations of the matrix

$$A = \begin{pmatrix} 1 & 2 & -1 \\ -2 & -4 & 3 \\ 0 & 1 & 5 \end{pmatrix}$$

Solution.

Factorization 1:

$$\begin{pmatrix} 1 & 2 & -1 \\ -2 & -4 & 3 \\ 0 & 1 & 5 \end{pmatrix} \xrightarrow{R_2+2R_1} \begin{pmatrix} 1 & 2 & -1 \\ 0 & 0 & 1 \\ 0 & 1 & 5 \end{pmatrix} \xrightarrow{R_2\leftrightarrow R_3} \begin{pmatrix} 1 & 2 & -1 \\ 0 & 1 & 5 \\ 0 & 0 & 1 \end{pmatrix} = U_1,$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{pmatrix} = L_1,$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} = P_1.$$

So, we have

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 2 & -1 \\ -2 & -4 & 3 \\ 0 & 1 & 5 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & -1 \\ 0 & 1 & 5 \\ 0 & 0 & 1 \end{pmatrix}.$$

Factorization 2:

$$\begin{pmatrix} 1 & 2 & -1 \\ -2 & -4 & 3 \\ 0 & 1 & 5 \end{pmatrix} \xrightarrow{R_1\leftrightarrow R_2} \begin{pmatrix} -2 & -4 & 3 \\ 1 & 2 & -1 \\ 0 & 1 & 5 \end{pmatrix} \xrightarrow{R_2+\frac{1}{2}R_1} \begin{pmatrix} -2 & -4 & 3 \\ 0 & 0 & \frac{1}{2} \\ 0 & 1 & 5 \end{pmatrix} \xrightarrow{R_2\leftrightarrow R_3} \begin{pmatrix} -2 & -4 & 3 \\ 0 & 1 & 5 \\ 0 & 0 & \frac{1}{2} \end{pmatrix} = U_2,$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 0 \\ -\frac{1}{2} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -\frac{1}{2} & 0 & 1 \end{pmatrix} = L_2,$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} = P_2.$$

So, we have

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 2 & -1 \\ -2 & -4 & 3 \\ 0 & 1 & 5 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -\frac{1}{2} & 0 & 1 \end{pmatrix} \begin{pmatrix} -2 & -4 & 3 \\ 0 & 1 & 5 \\ 0 & 0 & \frac{1}{2} \end{pmatrix}.$$

□