21-241: Matrix Algebra – Summer I, 2006

Practice Exam 3 Solutions

1. Write the quadratic form $p(x, y, z) = 9x^2 + 7y^2 + 11z^2 - 8xy + 8xz$ in matrix form. Classify the associated matrix. Does it allow a Cholesky factorization?

   SOLUTION. $p(x, y, z) = p(\mathbf{u}) = \mathbf{u}^T K \mathbf{u}$, where $\mathbf{u} = (x, y, z)^T$, $K = \begin{pmatrix} 9 & -4 & 4 \\ -4 & 7 & 0 \\ 4 & 0 & 11 \end{pmatrix}$. Apply Gaussian to reduce K to an upper triangular matrix:

$$\begin{pmatrix} 9 & -4 & 4 \\ -4 & 7 & 0 \\ 4 & 0 & 11 \end{pmatrix} \xrightarrow[R_3 - \frac{4}{9}R_1]{R_2 + \frac{4}{9}R_1} \begin{pmatrix} 9 & -4 & 4 \\ 0 & \frac{47}{9} & \frac{16}{9} \\ 0 & \frac{16}{9} & \frac{83}{9} \end{pmatrix} \xrightarrow{R_3 - \frac{16}{47}R_2} \begin{pmatrix} \boxed{9} & -4 & 4 \\ 0 & \boxed{\frac{47}{9}} & \frac{16}{9} \\ 0 & 0 & \boxed{\frac{405}{47}} \end{pmatrix}$$

   We see that K is regular and has all positive pivots. Hence it's positive definite and allows a Cholesky factorization. □

2. For the quadratic function $p(x, y, z) = x^2 + 2y^2 + z^2 - 2xy - 2yz - 2xz + 6x + 2y - 4z + 1$, determine if there is a minimum/maximum. If so, find the minimizer/maximizer and the minimum/maximum value of the function.

   SOLUTION. $p(x, y, z) = p(\mathbf{u}) = \mathbf{u}^T K \mathbf{u} - 2\mathbf{u}^T \mathbf{f} + c$, where $K = \begin{pmatrix} 1 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 1 \end{pmatrix}$, $\mathbf{f} = \begin{pmatrix} -3 \\ -1 \\ 2 \end{pmatrix}$, $c = 1$.
   Apply Gaussian to reduce K to an upper triangular matrix:

$$\begin{pmatrix} 1 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 1 \end{pmatrix} \xrightarrow[R_3 + R_1]{R_2 + R_1} \begin{pmatrix} 1 & -1 & -1 \\ 0 & 1 & -2 \\ 0 & -2 & 0 \end{pmatrix} \xrightarrow{R_3 + 2R_2} \begin{pmatrix} \boxed{1} & -1 & -1 \\ 0 & \boxed{1} & -2 \\ 0 & 0 & \boxed{-4} \end{pmatrix}$$

   We see that K has both positive and negative pivots. Hence it's indefinite and p has no minimum.

   To determine if p has a maximum, it's equivalent to check if $-p$ has a minimum. Since $-p(\mathbf{u}) = \mathbf{u}^T(-K)\mathbf{u} - 2\mathbf{u}^T(-\mathbf{f}) - c$, we apply Gaussian to $-K$:

$$-\begin{pmatrix} 1 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 1 \end{pmatrix} \xrightarrow[R_3 + R_1]{R_2 + R_1} -\begin{pmatrix} 1 & -1 & -1 \\ 0 & 1 & -2 \\ 0 & -2 & 0 \end{pmatrix} \xrightarrow{R_3 + 2R_2} -\begin{pmatrix} 1 & -1 & -1 \\ 0 & 1 & -2 \\ 0 & 0 & -4 \end{pmatrix} = \begin{pmatrix} \boxed{-1} & 1 & 1 \\ 0 & \boxed{-1} & 2 \\ 0 & 0 & \boxed{4} \end{pmatrix}$$

   We see that $-K$ also has both positive and negative pivots. (Actually it has to. Why?) Hence $-K$ is indefinite and $-p$ has no minimum. Therefore p has neither minimum nor maximum. □

3. Find an orthonormal basis for the column space of the matrix $\begin{pmatrix} 3 & -5 & 1 \\ 1 & 1 & 1 \\ -1 & 5 & -2 \\ 3 & -7 & 8 \end{pmatrix}$.

Solution. Denote the matrix by A. We first need to find a basis for $\operatorname{Col} A$ by reducing A in the echelon form.

$$\begin{pmatrix} 3 & -5 & 1 \\ 1 & 1 & 1 \\ -1 & 5 & -2 \\ 3 & -7 & 8 \end{pmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{pmatrix} 1 & 1 & 1 \\ 3 & -5 & 1 \\ -1 & 5 & -2 \\ 3 & -7 & 8 \end{pmatrix} \xrightarrow[R_4-3R_1]{\substack{R_2-3R_1 \\ R_3+R_1}} \begin{pmatrix} 1 & 1 & 1 \\ 0 & -8 & -2 \\ 0 & 6 & -1 \\ 0 & -10 & 5 \end{pmatrix} \xrightarrow[R_4-\frac{5}{4}R_2]{R_3+\frac{3}{4}R_2} \begin{pmatrix} \boxed{1} & 1 & 1 \\ 0 & \boxed{-8} & -2 \\ 0 & 0 & \boxed{-\frac{5}{2}} \\ 0 & 0 & \frac{15}{2} \end{pmatrix}$$

We see that each column contains a pivot, so the three columns of A, denoted by $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$, form a basis for $\operatorname{Col} A$. Apply Gram-Schmidt formula to convert this basis to an orthogonal one:

$$\mathbf{w}_1 = \mathbf{v}_1 = \begin{pmatrix} 3 \\ 1 \\ -1 \\ 3 \end{pmatrix},$$

$$\mathbf{w}_2 = \mathbf{v}_2 - \frac{\mathbf{v}_2 \cdot \mathbf{w}_1}{\|\mathbf{w}_1\|^2}\mathbf{w}_1 = \begin{pmatrix} -5 \\ 1 \\ 5 \\ -7 \end{pmatrix} - \frac{-40}{20}\begin{pmatrix} 3 \\ 1 \\ -1 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ 3 \\ -1 \end{pmatrix},$$

$$\mathbf{w}_3 = \mathbf{v}_3 - \frac{\mathbf{v}_3 \cdot \mathbf{w}_1}{\|\mathbf{w}_1\|^2}\mathbf{w}_1 - \frac{\mathbf{v}_3 \cdot \mathbf{w}_2}{\|\mathbf{w}_2\|^2}\mathbf{w}_2 = \begin{pmatrix} 1 \\ 1 \\ -2 \\ 8 \end{pmatrix} - \frac{30}{20}\begin{pmatrix} 3 \\ 1 \\ -1 \\ 3 \end{pmatrix} - \frac{-10}{20}\begin{pmatrix} 1 \\ 3 \\ 3 \\ -1 \end{pmatrix} = \begin{pmatrix} -3 \\ 1 \\ 1 \\ 3 \end{pmatrix}.$$

Finally, we normalize $\mathbf{w}_1$, $\mathbf{w}_2$, $\mathbf{w}_3$ to obtain an orthonormal basis:

$$\mathbf{u}_1 = \frac{\mathbf{w}_1}{\|\mathbf{w}_1\|} = \frac{1}{2\sqrt{5}}\begin{pmatrix} 3 \\ 1 \\ -1 \\ 3 \end{pmatrix}, \qquad \mathbf{u}_2 = \frac{\mathbf{w}_2}{\|\mathbf{w}_2\|} = \frac{1}{2\sqrt{5}}\begin{pmatrix} 1 \\ 3 \\ 3 \\ -1 \end{pmatrix}, \qquad \mathbf{u}_3 = \frac{\mathbf{w}_3}{\|\mathbf{w}_3\|} = \frac{1}{2\sqrt{5}}\begin{pmatrix} -3 \\ 1 \\ 1 \\ 3 \end{pmatrix}.$$

□

4. Show that the elementary row operation matrix corresponding to the interchange of two rows is an improper orthogonal matrix.

   Proof. Let E be an elementary matrix obtained by interchanging row i and row j of the identity matrix. Note that E can also be obtained by interchanging column i and column j of the identity. Thus the columns of E are still the standard basis, which is orthonormal. So E is orthogonal. Since interchanging two rows changes the sign of the determinant, $\det E = -\det I = -1$, which implies that E is an improper orthogonal matrix. □

5. Show that the matrix $\begin{pmatrix} 2 & 1 & -1 \\ 1 & 0 & 2 \\ 2 & -1 & 3 \end{pmatrix}$ is nonsingular. Find the QR factorization of the matrix.

   Solution. Denote the matrix by A. Then $\det A = 6$ (by the formula for 3×3 matrices). So A is nonsingular. Denote the columns of A by $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$. Apply Gram-Schmidt formula to get an orthogonal basis:

$$\mathbf{w}_1 = \mathbf{v}_1 = \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix},$$

$$\mathbf{w}_2 = \mathbf{v}_2 - \frac{\mathbf{v}_2 \cdot \mathbf{w}_1}{\|\mathbf{w}_1\|^2}\mathbf{w}_1 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} - \frac{0}{9}\begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix},$$

$$\mathbf{w}_3 = \mathbf{v}_3 - \frac{\mathbf{v}_3 \cdot \mathbf{w}_1}{\|\mathbf{w}_1\|^2}\mathbf{w}_1 - \frac{\mathbf{v}_3 \cdot \mathbf{w}_2}{\|\mathbf{w}_2\|^2}\mathbf{w}_2 = \begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix} - \frac{6}{9}\begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} - \frac{-4}{2}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} -\frac{1}{3} \\ \frac{4}{3} \\ -\frac{1}{3} \end{pmatrix}.$$

Then we normalize $\mathbf{w}_1$, $\mathbf{w}_2$, $\mathbf{w}_3$ to obtain an orthonormal basis:

$$\mathbf{u}_1 = \frac{\mathbf{w}_1}{\|\mathbf{w}_1\|} = \begin{pmatrix} \frac{2}{3} \\ \frac{1}{3} \\ \frac{2}{3} \end{pmatrix}, \qquad \mathbf{u}_2 = \frac{\mathbf{w}_2}{\|\mathbf{w}_2\|} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ -\frac{1}{\sqrt{2}} \end{pmatrix}, \qquad \mathbf{u}_3 = \frac{\mathbf{w}_3}{\|\mathbf{w}_3\|} = \begin{pmatrix} -\frac{1}{3\sqrt{2}} \\ \frac{4}{3\sqrt{2}} \\ -\frac{1}{3\sqrt{2}} \end{pmatrix}.$$

Thus we obtain the orthogonal matrix Q by combining $\mathbf{u}_1$, $\mathbf{u}_2$, $\mathbf{u}_3$ in a single matrix

$$Q = \begin{pmatrix} \frac{2}{3} & \frac{1}{\sqrt{2}} & -\frac{1}{3\sqrt{2}} \\ \frac{1}{3} & 0 & \frac{4}{3\sqrt{2}} \\ \frac{2}{3} & -\frac{1}{\sqrt{2}} & -\frac{1}{3\sqrt{2}} \end{pmatrix}.$$

The upper triangular matrix R equals

$$R = Q^T A = \begin{pmatrix} \frac{2}{3} & \frac{1}{3} & \frac{2}{3} \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ -\frac{1}{3\sqrt{2}} & \frac{4}{3\sqrt{2}} & -\frac{1}{3\sqrt{2}} \end{pmatrix}\begin{pmatrix} 2 & 1 & -1 \\ 1 & 0 & 2 \\ 2 & -1 & 3 \end{pmatrix} = \begin{pmatrix} 3 & 0 & 2 \\ 0 & \sqrt{2} & -2\sqrt{2} \\ 0 & 0 & \sqrt{2} \end{pmatrix}.$$

□

6. Find the closest point to $\mathbf{b} = (1,0,2)^T$ in the subspace $W = \operatorname{span}\{(1,-1,1)^T, (1,2,-1)^T\}$.

SOLUTION. Let $\mathbf{v}_1 = (1,-1,1)^T$, $\mathbf{v}_2 = (1,2,-1)^T$. Apply the Gram-Schmidt formula,

$$\mathbf{w}_1 = \mathbf{v}_1 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix},$$

$$\mathbf{w}_2 = \mathbf{v}_2 - \frac{\mathbf{v}_2 \cdot \mathbf{w}_1}{\|\mathbf{w}_1\|^2}\mathbf{w}_1 = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} - \frac{-2}{3}\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{5}{3} \\ \frac{4}{3} \\ -\frac{1}{3} \end{pmatrix}.$$

The closest point to $\mathbf{b}$ is the orthogonal projection of $\mathbf{b}$ onto W:

$$\operatorname{proj}_W \mathbf{b} = \frac{\mathbf{b} \cdot \mathbf{w}_1}{\|\mathbf{w}_1\|^2}\mathbf{w}_1 + \frac{\mathbf{b} \cdot \mathbf{w}_2}{\|\mathbf{w}_2\|^2}\mathbf{w}_2 = \frac{3}{3}\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} + \frac{1}{14/3}\begin{pmatrix} \frac{5}{3} \\ \frac{4}{3} \\ -\frac{1}{3} \end{pmatrix} = \begin{pmatrix} \frac{19}{14} \\ -\frac{10}{14} \\ \frac{13}{14} \end{pmatrix}.$$

□

7. Let $W = \ker \begin{pmatrix} 1 & 2 & -1 \\ 2 & 0 & 2 \end{pmatrix}$, $\mathbf{v} = (1,1,0)^T$. Decompose $\mathbf{v}$ with respect to W as $\mathbf{v} = \mathbf{w} + \mathbf{z}$, where $\mathbf{w} \in W$, $\mathbf{z} \in W^{\perp}$.

SOLUTION. We first need find a basis for W. Reduce the matrix in the echelon form:

$$\begin{pmatrix} 1 & 2 & -1 \\ 2 & 0 & 2 \end{pmatrix} \xrightarrow{R_2 - 2R_1} \begin{pmatrix} \boxed{1} & 2 & -1 \\ 0 & \boxed{\text{-4}} & 4 \end{pmatrix}.$$

We see that x_3 is free, and by back substitution, $x_2 = x_3$, $x_1 = -x_3$. Then the general solution to the homogeneous system is

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -x_3 \\ x_3 \\ x_3 \end{pmatrix} = x_3 \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}.$$

So $\mathbf{w}_1 = (-1,1,1)^T$ forms an (orthogonal) basis for W. Then

$$\mathbf{w} = \operatorname{proj}_W \mathbf{v} = \frac{\mathbf{v} \cdot \mathbf{w}_1}{\|\mathbf{w}_1\|^2} \mathbf{w}_1 = \mathbf{0},$$

$$\mathbf{z} = \mathbf{v} - \mathbf{w} = \mathbf{v} = (1,1,0)^T.$$

□