21-241: Matrix Algebra – Summer I, 2006

Practice Exam 2

1. Let $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$, $\mathbf{v}_2 = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}$, $\mathbf{v}_3 = \begin{pmatrix} 4 \\ 2 \\ 6 \end{pmatrix}$, and $\mathbf{w} = \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix}$.

   (a) Is $\mathbf{w}$ in $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$? How many vectors are in $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$?

   (b) How many vectors are in span $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$?

   (c) Is $\mathbf{w}$ in the subspace spanned by $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$? Why?

   SOLUTION.

   (a) No. $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a set containing only three vectors $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$. Apparently, $\mathbf{w}$ equals none of these three, so $\mathbf{w} \notin \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$.

   (b) span $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is the set containing ALL possible linear combinations of $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$. Particularly, any scalar multiple of $\mathbf{v}_1$, say, $2\mathbf{v}_1, 3\mathbf{v}_1, 4\mathbf{v}_1, \cdots$, are all in the span. This implies span $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ contains infinitely many vectors.

   (c) To determine whether $\mathbf{w}$ belongs to span $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$, we are to look to write $\mathbf{w}$ as a linear combination of $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$. For this purpose, we need to find three scalars c_1, c_2, c_3, such that $\mathbf{w} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3$. This amounts to solve the system $A\mathbf{c} = \mathbf{w}$ for $\mathbf{c} = (c_1, c_2, c_3)^T$, where matrix $A = (\mathbf{v}_1\ \mathbf{v}_2\ \mathbf{v}_3)$. Note that actually we only need to determine if this system allows a solution. Now apply Gaussian to reduce the augmented matrix in the echelon form:

   $$\left(\begin{array}{ccc|c} 1 & 2 & 4 & 3 \\ 0 & 1 & 2 & 1 \\ -1 & 3 & 6 & 2 \end{array}\right) \xrightarrow{R_3+R_1} \left(\begin{array}{ccc|c} 1 & 2 & 4 & 3 \\ 0 & 1 & 2 & 1 \\ 0 & 5 & 10 & 5 \end{array}\right) \xrightarrow{R_3-5R_2} \left(\begin{array}{ccc|c} \boxed{1} & 2 & 4 & 3 \\ 0 & \boxed{1} & 2 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

   The bottom row doesn't lead to inconsistency, so the system allows a solution (actually has infinitely many). This shows that $\mathbf{w}$ is in the subspace spanned by $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$. $\square$

2. Given subspaces H and K of a vector space V, the **sum** of H and K, written as $H + K$, is the set of all vectors in V that can be written as the sum of two vectors, one in H and the other in K; that is,

   $$H + K = \{\mathbf{w} | \mathbf{w} = \mathbf{u} + \mathbf{v} \text{ for some } \mathbf{u} \in H \text{ and some } \mathbf{v} \in K\}$$

   (a) Show that $H + K$ is subspace of V.

   (b) Show that H is a subspace of $H + K$ and K is a subspace of $H + K$.

   PROOF.

   (a) Since H and K are subspaces of V, the zero vector $\mathbf{0}$ has to belong to them both. Taking $\mathbf{u} = \mathbf{v} = \mathbf{0}$, we have $\mathbf{w} = \mathbf{0} + \mathbf{0} = \mathbf{0}$, which, by definition, belongs to $H + K$. Next, we are to show $H + K$ is closed under both addition and scalar multiplication. Suppose $\mathbf{w}_1, \mathbf{w}_2$ are two vectors in $H + K$. By definition, they can be written as

   $$\mathbf{w}_1 = \mathbf{u}_1 + \mathbf{v}_1, \quad \mathbf{w}_2 = \mathbf{u}_2 + \mathbf{v}_2, \quad \text{for some } \mathbf{u}_1, \mathbf{u}_2 \in H \text{ and some } \mathbf{v}_1, \mathbf{v}_2 \in K.$$

   Hence,

   $$\mathbf{w}_1 + \mathbf{w}_2 = (\mathbf{u}_1 + \mathbf{v}_1) + (\mathbf{u}_2 + \mathbf{v}_2) = (\mathbf{u}_1 + \mathbf{u}_2) + (\mathbf{v}_1 + \mathbf{v}_2),$$

where, $\mathbf{u}_1 + \mathbf{u}_2 \in H$ because H is a subspace, thus closed under addition; and $\mathbf{v}_1 + \mathbf{v}_2 \in K$ similarly. This shows that $\mathbf{w}_1 + \mathbf{w}_2$ can be written as the sum of two vectors, one in H and the other in K. So, again by definition, $\mathbf{w}_1 + \mathbf{w}_2 \in H + K$, namely, $H + K$ is closed under addition. For scalar multiplication, note that given scalar c,

$$c\mathbf{w}_1 = c(\mathbf{u}_1 + \mathbf{v}_1) = c\mathbf{u}_1 + c\mathbf{v}_1,$$

where $c\mathbf{u}_1 \in H$ because H is closed under scalar multiplication; and $c\mathbf{v}_1 \in K$ parallelly. Now that $c\mathbf{w}_1$ has been written as the sum of two vectors, one in H and the other in K, it's in $H+K$. That is, $H + K$ is closed under scalar multiplication. And we are done.

(b) Since H is a subspace of V, it's nonempty, closed under addition and scalar multiplication. We only need to show that H is a subset of $H + K$. This is derived from the fact that each vector in H can be written as the sum of itself, which belongs to H, and the zero vector, which belongs to K. A similar argument justifies K is a subspace of $H + K$, too. □

3. Let $\mathbf{x}$ and $\mathbf{y}$ be linearly independent elements of a vector space V. Show that $\mathbf{u} = a\mathbf{x} + b\mathbf{y}$ and $\mathbf{v} = c\mathbf{x} + d\mathbf{y}$ are linearly independent if and only if $ad - bc \neq 0$. Is the entire collection $\mathbf{x}$, $\mathbf{y}$, $\mathbf{u}$,$\mathbf{v}$ linearly independent?

Proof. Let $A = (\mathbf{x}\ \mathbf{y})$, $B = (\mathbf{u}\ \mathbf{v})$, $C = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$, then

$$AC = (\mathbf{x}\ \mathbf{y}) \begin{pmatrix} a & c \\ b & d \end{pmatrix} = (a\mathbf{x} + b\mathbf{y} \quad c\mathbf{x} + d\mathbf{y}) = (\mathbf{u}\ \mathbf{v}) = B.$$

Two key facts we'll use later are that $\mathbf{u}$ and $\mathbf{v}$ (or, $\mathbf{x}$ and $\mathbf{y}$) are linearly independent if and only if the homogeneous system $B\mathbf{r} = \mathbf{0}$ (or, $A\mathbf{r} = \mathbf{0}$) allows only trivial solution, denoted by Fact 1 (or, Fact 2). Now slow down, carefully think of the following deduction process, and make sure you really understand each step involved.

$$\begin{aligned}
&\mathbf{u}, \mathbf{v} \text{ are linearly independent} && \\
\Longleftrightarrow\ & B\mathbf{r} = \mathbf{0} \text{ has only trivial solution} && \text{(by Fact 1)} \\
\Longleftrightarrow\ & (AC)\mathbf{r} = \mathbf{0} \text{ has only trivial solution} && \text{(since } AC = B\text{)} \\
\Longleftrightarrow\ & A(C\mathbf{r}) = \mathbf{0} \text{ has only trivial solution} && \text{(by associativity)} \\
\Longleftrightarrow\ & C\mathbf{r} = \mathbf{0} \text{ has only trivial solution} && \text{(by Fact 2, replace } \mathbf{r} \text{ by } C\mathbf{r}\text{)} \\
\Longleftrightarrow\ & C \text{ is nonsingular} && \\
\Longleftrightarrow\ & \det C \neq 0 && \\
\Longleftrightarrow\ & ad - bc \neq 0 &&
\end{aligned}$$

The entire collection $\mathbf{x}$, $\mathbf{y}$, $\mathbf{u}$,$\mathbf{v}$ is linearly dependent, since we have four scalars, $a, b, -1, 0$, not all zero, such that the linear combination $a\mathbf{x} + b\mathbf{y} + (-1)\mathbf{u} + 0\mathbf{v} = \mathbf{0}$. □

4. Find bases for the column space (range) and null space (kernel) of the matrix $A = \begin{pmatrix} -2 & 4 & -2 & -4 \\ 2 & -6 & -3 & 1 \\ -3 & 8 & 2 & -3 \end{pmatrix}$.

Solution. To find the basis for column space, we need to find pivot column(s). To find the basis for null space, we need to find general solution to the homogeneous system $A\mathbf{x} = \mathbf{0}$. Both can be achieved by reducing the matrix in the echelon form.

$$\begin{pmatrix} -2 & 4 & -2 & -4 \\ 2 & -6 & -3 & 1 \\ -3 & 8 & 2 & -3 \end{pmatrix} \xrightarrow[R_3-\frac{3}{2}R_1]{R_2+R_1} \begin{pmatrix} -2 & 4 & -2 & -4 \\ 0 & -2 & -5 & -3 \\ 0 & 2 & 5 & 3 \end{pmatrix} \xrightarrow{R_3+R_2} \begin{pmatrix} \boxed{-2} & 4 & -2 & -4 \\ 0 & \boxed{-2} & -5 & -3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

We see that the first two columns are pivot columns, so the first two column of the ORIGINAL MATRIX A, namely, $\{(-2,2,-3)^T, (4,-6,8)^T\}$, form a basis for $\operatorname{Col} A$. The last two columns are free, and we can easily read the general solution from the echelon form:

$$x_2 = -\frac{5}{2}x_3 - \frac{3}{2}x_4, \quad x_1 = 2x_2 - x_3 - 2x_4 = -6x_3 - 5x_4, \quad x_3, x_4 \text{ free}$$

Written in vector form,

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -6x_3 - 5x_4 \\ -\frac{5}{2}x_3 - \frac{3}{2}x_4 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -6x_3 \\ -\frac{5}{2}x_3 \\ x_3 \\ 0 \end{pmatrix} + \begin{pmatrix} -5x_4 \\ -\frac{3}{2}x_4 \\ 0 \\ x_4 \end{pmatrix} = x_3 \begin{pmatrix} -6 \\ \frac{5}{2} \\ 1 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} -5 \\ -\frac{3}{2} \\ 0 \\ 1 \end{pmatrix}$$

Thus, $\{(-6, \frac{5}{2}, 1, 0)^T, (-5, \frac{3}{2}, 0, 1)^T\}$ form a basis for $\operatorname{Nul} A$. □

5. Show that $\left\{\mathbf{u}_1 = \begin{pmatrix} 3 \\ -5 \end{pmatrix}, \mathbf{u}_2 = \begin{pmatrix} -4 \\ 6 \end{pmatrix}\right\}$ is a basis for $\mathbb{R}^2$. Let $\mathbf{x} = \begin{pmatrix} 2 \\ -6 \end{pmatrix}$. Find the coordinate vector for $\mathbf{x}$ with respect to this basis.

SOLUTION. First of all, $\mathbf{u}_1$ and $\mathbf{u}_2$ are linearly independent because they are not multiples of each other. Next, we are to characterize vectors in $\operatorname{span}\{\mathbf{u}_1, \mathbf{u}_2\}$. Suppose vector $\mathbf{b} \in \mathbb{R}^2$ belongs to $\operatorname{span}\{\mathbf{u}_1, \mathbf{u}_2\}$, then the linear system $A\mathbf{y} = \mathbf{b}$ is consistent, where matrix $A = (\mathbf{u}_1 \quad \mathbf{u}_2)$. Applying Gaussian to the augmented matrix, we get

$$\left(\begin{array}{cc|c} 3 & -4 & b_1 \\ -5 & 6 & b_2 \end{array}\right) \xrightarrow{R_2+\frac{5}{3}R_1} \left(\begin{array}{cc|c} \boxed{3} & -4 & b_1 \\ 0 & \boxed{-\frac{2}{3}} & b_2 + \frac{5}{3}b_1 \end{array}\right)$$

The system has a pivot in each row, thus is always consistent for all possible $\mathbf{b} \in \mathbb{R}^2$. Therefore, $\operatorname{span}\{\mathbf{u}_1, \mathbf{u}_2\} = \mathbb{R}^2$, and $\{\mathbf{u}_1, \mathbf{u}_2\}$ form a basis for $\mathbb{R}^2$. To find the coordinate vector for $\mathbf{x}$, we need find the solution to $A\mathbf{y} = \mathbf{x}$. Replacing b_1, b_2 by 2, -6 respectively in the echelon form we obtained above, we can write out the solution $\mathbf{y} = (6,4)^T$. This is to say, $\mathbf{x} = 6\mathbf{u}_1 + 4\mathbf{u}_2$, so the coordinate vector for $\mathbf{x}$ w.r.t $\{\mathbf{u}_1, \mathbf{u}_2\}$ is $(6,4)^T$. □

6. Let V be an inner product space.

   (a) Prove that $\langle \mathbf{x}, \mathbf{v} \rangle = 0$ for all $\mathbf{v} \in V$ if and only if $\mathbf{x} = \mathbf{0}$.

   (b) Prove that $\langle \mathbf{x}, \mathbf{v} \rangle = \langle \mathbf{y}, \mathbf{v} \rangle$ for all $\mathbf{v} \in V$ if and only if $\mathbf{x} = \mathbf{y}$.

   (c) Let $\mathbf{v}_1, \cdots, \mathbf{v}_n$ be a basis for V. Prove that $\langle \mathbf{x}, \mathbf{v}_i \rangle = \langle \mathbf{y}, \mathbf{v}_i \rangle$, $i = 1, \cdots, n$, if and only if $\mathbf{x} = \mathbf{y}$.

PROOF.

(a) Suppose $\langle \mathbf{x}, \mathbf{v} \rangle = 0$ for all $\mathbf{v} \in V$. Simply let $\mathbf{v} = \mathbf{x}$ and we get $\langle \mathbf{x}, \mathbf{x} \rangle = 0$, which implies $\mathbf{x} = 0$.

(b) We reduce the equivalence as follows:

$$\begin{aligned}
&\langle \mathbf{x}, \mathbf{v}\rangle = \langle \mathbf{y}, \mathbf{v}\rangle, \forall\ \mathbf{v} \in V \\
\iff &\langle \mathbf{x}, \mathbf{v}\rangle - \langle \mathbf{y}, \mathbf{v}\rangle = 0, \forall\ \mathbf{v} \in V \\
\iff &\langle \mathbf{x} - \mathbf{y}, \mathbf{v}\rangle = 0, \forall\ \mathbf{v} \in V && \text{(by bilinearity)} \\
\iff &\mathbf{x} - \mathbf{y} = \mathbf{0} && \text{(by part (a))} \\
\iff &\mathbf{x} = \mathbf{y}
\end{aligned}$$

(c) If $\mathbf{x} = \mathbf{y}$, of course we have $\langle \mathbf{x}, \mathbf{v}_i\rangle = \langle \mathbf{y}, \mathbf{v}_i\rangle$, $i = 1, 2, \cdots, n$. Reversely, suppose $\langle \mathbf{x}, \mathbf{v}_i\rangle = \langle \mathbf{y}, \mathbf{v}_i\rangle$, $i = 1, \cdots, n$. Since $\mathbf{v}_1, \cdots, \mathbf{v}_n$ is a basis for V, any vector $\mathbf{v} \in V$ can be written as a linear combination of the n vectors, say, $\mathbf{v} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n$. Linearity of inner product implies

$$\begin{aligned}
\langle \mathbf{x}, \mathbf{v}\rangle &= \langle \mathbf{x}, c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n\rangle \\
&= c_1\langle \mathbf{x}, \mathbf{v}_1\rangle + c_2\langle \mathbf{x}, \mathbf{v}_2\rangle + \cdots + c_n\langle \mathbf{x}, \mathbf{v}_n\rangle && \text{(by linearity)} \\
&= c_1\langle \mathbf{y}, \mathbf{v}_1\rangle + c_2\langle \mathbf{y}, \mathbf{v}_2\rangle + \cdots + c_n\langle \mathbf{y}, \mathbf{v}_n\rangle && \text{(since } \langle \mathbf{x}, \mathbf{v}_i\rangle = \langle \mathbf{y}, \mathbf{v}_i\rangle\text{)} \\
&= \langle \mathbf{y}, c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n\rangle && \text{(by linearity)} \\
&= \langle \mathbf{y}, \mathbf{v}\rangle
\end{aligned}$$

Since this equality holds for all $\mathbf{v} \in V$, part (b) tells us that $\mathbf{x} = \mathbf{y}$. □

7. Prove that

$$(a_1 + a_2 + \cdots + a_n)^2 \leqslant n(a_1^2 + a_2^2 + \cdots + a_n^2)$$

for any real numbers $a_1, \cdots, a_n$. When does equality hold?

PROOF. Let $\mathbf{u} = (a_1, a_2, \cdots, a_n)^T$, $\mathbf{v} = (1, 1, \cdots, 1)^T$. Then,

$$\mathbf{u} \cdot \mathbf{v} = a_1 + a_2 + \cdots + a_n, \quad \|\mathbf{u}\|^2 = a_1^2 + a_2^2 + \cdots + a_n^2, \quad \|\mathbf{v}\|^2 = n.$$

By Cauchy-Schwarz inequality, $|\mathbf{u} \cdot \mathbf{v}| \leqslant \|\mathbf{u}\|\,\|\mathbf{v}\|$. Squaring both sides, we obtain

$$(a_1 + a_2 + \cdots + a_n)^2 \leqslant n(a_1^2 + a_2^2 + \cdots + a_n^2).$$

This completes the proof. □

8. Verify the formula $\|\mathbf{v}\| = \max\{|v_1 + v_2|, |v_1 - v_2|\}$ defines a norm on $\mathbb{R}^2$. Establish the equivalence between this norm and the usual Euclidean norm $\|\cdot\|_2$.

PROOF. We need verify positivity, homogeneity and triangle inequality one by one.

Positivity: Since $|v_1 + v_2| \geqslant 0$, $|v_1 - v_2| \geqslant 0$, it's clear that $\|\mathbf{v}\| \geqslant 0$. Moreover, $\|\mathbf{v}\| = 0$ if and only if $|v_1 + v_2| = |v_1 - v_2| = 0$, that is, $v_1 = v_2 = 0$, namely $\mathbf{v} = \mathbf{0}$.

Homogeneity: $\|c\mathbf{v}\| = \max\{|cv_1 + cv_2|, |cv_1 - cv_2|\} = |c| \max\{|v_1 + v_2|, |v_1 - v_2|\} = |c|\,\|\mathbf{v}\|$.

Triangle Inequality: We need use triangle inequality for absolute value

$$|a + b| \leqslant |a| + |b|,$$

and the fact (denoted by Fact 3) that "the maximum of sums is less than or equal to the sum of maximums",

$$\max\{a_1 + a_2, b_1 + b_2\} \leqslant \max\{a_1, b_1\} + \max\{a_2, b_2\}.$$

Both inequalities are usual. Having them, we can obtain the triangle inequality for norm $\|\cdot\|$:

$$\begin{aligned}\|\mathbf{u}+\mathbf{v}\| &= \max\{|(u_1+v_1)+(u_2+v_2)|, |(u_1+v_1)-(u_2+v_2)|\} && \text{(by definition)}\\ &\leqslant \max\{|u_1+u_2|+|v_1+v_2|, |u_1-u_2|+|v_1-v_2|\} && \text{(by triangle inequality)}\\ &\leqslant \max\{|u_1+u_2|, |u_1-u_2|\}+\max\{|v_1+v_2|, |v_1-v_2|\} && \text{(by Fact 3)}\\ &= \|\mathbf{u}\|+\|\mathbf{v}\|. && \text{(by definition)}\end{aligned}$$

We just directly proved $\|\cdot\|$ defines a norm. If we think in another way, it can be verified that

$$\|\mathbf{v}\| = \max\{|v_1+v_2|, |v_1-v_2|\} = |v_1|+|v_2| = \|\mathbf{v}\|_1,$$

namely, $\|\cdot\|$ is actually the 1-norm $\|\cdot\|_1$. The proof is not difficult and left to you.

To show the equivalence of two norms, we need to find two POSITIVE constants m, M, such that

$$m\|\mathbf{v}\|_2 \leqslant \|\mathbf{v}\| \leqslant M\|\mathbf{v}\|_2, \quad \text{for all } \mathbf{v}\in\mathbb{R}^2.$$

You may already find it's convenient to compare squares of norms when Euclidean norm is involved. So, let's square:

$$\begin{aligned}\|\mathbf{v}\|^2 &= (\max\{|v_1+v_2|,\ |v_1-v_2|\})^2 = \max\{|v_1+v_2|^2,\ |v_1-v_2|^2\}\\ &= \max\{v_1^2+v_2^2+2v_1v_2,\ v_1^2+v_2^2-2v_1v_2\} = v_1^2+v_2^2+\max\{2v_1v_2,\ -2v_1v_2\}\\ &= v_1^2+v_2^2+|2v_1v_2| \geqslant v_1^2+v_2^2 = \|\mathbf{v}\|_2^2.\end{aligned}$$

This allows us letting $m=1$. On the other hand, since $|2v_1v_2| \leqslant v_1^2+v_2^2$,

$$\|\mathbf{v}\|^2 = v_1^2+v_2^2+|2v_1v_2| \leqslant 2(v_1^2+v_2^2) = 2\|\mathbf{v}\|_2^2.$$

Therefore, we can choose $M=\sqrt{2}$. Thus we complete the proof. □

9. Prove the matrix $A = \begin{pmatrix} 1 & 1 & 1\\ 1 & 2 & -2\\ 1 & -2 & 14\end{pmatrix}$ is positive definite. Find its Cholesky factorization.

Proof. We apply Gaussian to show the matrix has all positive pivots and find the LDL^T factorization.

$$\begin{pmatrix} 1 & 1 & 1\\ 1 & 2 & -2\\ 1 & -2 & 14\end{pmatrix} \xrightarrow[R_3-R_1]{R_2-R_1} \begin{pmatrix} 1 & 1 & 1\\ 0 & 1 & -3\\ 0 & -3 & 13\end{pmatrix} \xrightarrow{R_3+3R_2} \begin{pmatrix} \boxed{1} & 1 & 1\\ 0 & \boxed{1} & -3\\ 0 & 0 & \boxed{4}\end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix} \xrightarrow[R_3+R_1]{R_2+R_1} \begin{pmatrix} 1 & 0 & 0\\ 1 & 1 & 0\\ 1 & 0 & 1\end{pmatrix} \xrightarrow{R_3-3R_2} \begin{pmatrix} 1 & 0 & 0\\ 1 & 1 & 0\\ 1 & -3 & 1\end{pmatrix} = L$$

Now, we see the matrix is regular and has all positive pivots $1, 1, 4$, thus is positive definite. Let $D = \operatorname{diag}(1,1,4)$, $S = \operatorname{diag}(1,1,2)$, then we obtain the Cholesky factorization

$$A = LDL^T = LS^2L^T = LSS^TL^T = MM^T,$$

where $M = LS = \left(\begin{smallmatrix} 1 & 0 & 0\\ 1 & 1 & 0\\ 1 & -3 & 2\end{smallmatrix}\right)$. □