21-241: Matrix Algebra – Summer I, 2006

Exam 3 Solutions

Note: For all problems in this exam (except No. 3) use the Euclidean dot product and norm on $\mathbb{R}^n$.

1. (18 points) True or False. (Don't need to justify)

   (a) If $\mathbf{x}$ is not in a subspace W, then $\mathbf{x} - \text{proj}_W \mathbf{x}$ is not zero.

   SOLUTION. **True**. If $\mathbf{x} - \text{proj}_W \mathbf{x} = \mathbf{0}$, then $\mathbf{x} = \text{proj}_W \mathbf{x} \in W$.

   (b) A matrix whose rows form an orthonormal basis for $\mathbb{R}^n$ is an orthogonal matrix.

   SOLUTION. **True**. If A's rows form an orthonormal basis, so do A^T's columns. Thus A^T is an orthogonal matrix, which implies $(A^T)^T = (A^T)^{-1} = (A^{-1})^T$. So $A^T = A^{-1}$, A is orthogonal.

   (c) An indefinite quadratic form is either positive semidefinite or negative semidefinite.

   SOLUTION. **False**. An indefinite quadratic form has both positive and negative pivots. It's neither positive semidefinite nor negative semidefinite.

   (d) $\mathbb{R}$ has only one orthonormal basis.

   SOLUTION. **False**. $\mathbb{R}$ has two orthonormal bases, 1 and -1.

   (e) A least-squares solution of $A\mathbf{x} = \mathbf{b}$ is a vector $\hat{\mathbf{x}}$ such that $\|\mathbf{b} - A\mathbf{x}\| \leqslant \|\mathbf{b} - A\hat{\mathbf{x}}\|$ for all $\mathbf{x}$ in $\mathbb{R}^n$.

   SOLUTION. **False**. The inequality sign should be "$\geqslant$".

   (f) Let V be an inner product space and W be an subspace of V, then $W = (W^\perp)^\perp$.

   SOLUTION. **True**. $(W^\perp)^\perp = \{\mathbf{v} \in V | \langle \mathbf{v}, \mathbf{w} \rangle = 0, \forall \mathbf{w} \in W^\perp\} \supseteq W$, and $\dim(W^\perp)^\perp = \dim V - \dim W^\perp = \dim W$, So $(W^\perp)^\perp = W$. □

2. (18 points) For the quadratic function $p(x, y, z) = -x^2 - 5y^2 - 4z^2 - 2xy - 2yz + 2xz + 2x + 4z + 5$, determine if there is a maximum. If so, find the maximizer and the maximum value of the function.

   SOLUTION. We consider $-p(x, y, z) = x^2 + 5y^2 + 4z^2 + 2xy + 2yz - 2xz - 2x - 4z - 5$ instead. $-p$ can be written as $-p(\mathbf{u}) = \mathbf{u}^T K \mathbf{u} - 2\mathbf{u}^T \mathbf{f} + c$, where

$$\mathbf{u} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \qquad K = \begin{pmatrix} 1 & 1 & -1 \\ 1 & 5 & 1 \\ -1 & 1 & 4 \end{pmatrix}, \qquad \mathbf{f} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}, \qquad c = -5.$$

   Apply Gaussian to the augmented matrix $(K|\mathbf{f})$:

$$\left(\begin{array}{ccc|c} 1 & 1 & -1 & 1 \\ 1 & 5 & 1 & 0 \\ -1 & 1 & 4 & 2 \end{array}\right) \xrightarrow[R_3+R_1]{R_2-R_1} \left(\begin{array}{ccc|c} 1 & 1 & -1 & 1 \\ 0 & 4 & 2 & -1 \\ 0 & 2 & 3 & 3 \end{array}\right) \xrightarrow{R_3-\frac{1}{2}R_2} \left(\begin{array}{ccc|c} \boxed{1} & 1 & -1 & 1 \\ 0 & \boxed{4} & 2 & -1 \\ 0 & 0 & \boxed{2} & \frac{7}{2} \end{array}\right)$$

   We see that K has all positive pivots, so $-p$ has a minimum, namely p has a maximum. By back substitution, the maximizer is $\mathbf{u}^* = (\frac{31}{8}, -\frac{9}{8}, \frac{7}{4})^T$. The maximum value is $p(\mathbf{u}^*) = -(c - (\mathbf{u}^*)^T \mathbf{f}) = \frac{99}{8}$. Note that here you don't have to compute $p(\mathbf{u}^*)$ using its definition, which may involve a lot of computational difficulties. □

3. (16 points) Construct an orthonormal basis of $\mathbb{R}^2$ for the non-standard inner product $\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x}^T \begin{pmatrix} 3 & 0 \\ 0 & 5 \end{pmatrix} \mathbf{y}$.

SOLUTION. The given inner product is $\langle \mathbf{x}, \mathbf{y} \rangle = (x_1\ x_2) \begin{pmatrix} 3 & 0 \\ 0 & 5 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = 3x_1y_1 + 5x_2y_2$. The orthonormal basis is not unique. We construct one starting with $\mathbf{v}_1 = (1, 0)^T$, $\mathbf{v}_2 = (0, 1)^T$. By Gram-Schmidt formula,

$$\mathbf{w}_1 = \mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad \|\mathbf{w}_1\| = \sqrt{3},$$

$$\mathbf{w}_2 = \mathbf{v}_2 - \frac{\langle \mathbf{v}_2, \mathbf{w}_1 \rangle}{\|\mathbf{w}_1\|^2} \mathbf{w}_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} - \frac{0}{3} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \qquad \|\mathbf{w}_2\| = \sqrt{5}.$$

After normalization, we get an orthonormal basis $\{\mathbf{u}_1, \mathbf{u}_2\}$ as follows:

$$\mathbf{u}_1 = \frac{\mathbf{w}_1}{\|\mathbf{w}_1\|} = \begin{pmatrix} \frac{1}{\sqrt{3}} \\ 0 \end{pmatrix}, \qquad \mathbf{u}_2 = \frac{\mathbf{w}_2}{\|\mathbf{w}_2\|} = \begin{pmatrix} 0 \\ \frac{1}{\sqrt{5}} \end{pmatrix}.$$

□

4. (15 points) Show that the matrix $\begin{pmatrix} 3 & 0 & 2 \\ 0 & 4 & 1 \\ -1 & 0 & 1 \end{pmatrix}$ is nonsingular. Find its QR factorization.

SOLUTION. Denote the matrix by A. Since $\det A = 12 - (-8) = 20$, A is nonsingular. Denote the columns of A by $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$. By Gram-Schmidt formula,

$$\mathbf{w}_1 = \mathbf{v}_1 = \begin{pmatrix} 3 \\ 0 \\ -1 \end{pmatrix}, \qquad \|\mathbf{w}_1\| = \sqrt{10},$$

$$\mathbf{w}_2 = \mathbf{v}_2 - \frac{\langle \mathbf{v}_2, \mathbf{w}_1 \rangle}{\|\mathbf{w}_1\|^2} \mathbf{w}_1 = \begin{pmatrix} 0 \\ 4 \\ 0 \end{pmatrix} - \frac{0}{10} \begin{pmatrix} 3 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 4 \\ 0 \end{pmatrix}, \qquad \|\mathbf{w}_2\| = 4,$$

$$\mathbf{w}_3 = \mathbf{v}_3 - \frac{\langle \mathbf{v}_3, \mathbf{w}_1 \rangle}{\|\mathbf{w}_1\|^2} \mathbf{w}_1 - \frac{\langle \mathbf{v}_3, \mathbf{w}_2 \rangle}{\|\mathbf{w}_2\|^2} \mathbf{w}_2 = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} - \frac{5}{10} \begin{pmatrix} 3 \\ 0 \\ -1 \end{pmatrix} - \frac{4}{16} \begin{pmatrix} 0 \\ 4 \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ 0 \\ \frac{3}{2} \end{pmatrix}, \qquad \|\mathbf{w}_3\| = \frac{\sqrt{10}}{2}.$$

After normalization, we get an orthonormal basis:

$$\mathbf{u}_1 = \frac{\mathbf{w}_1}{\|\mathbf{w}_1\|} = \begin{pmatrix} \frac{3}{\sqrt{10}} \\ 0 \\ -\frac{1}{\sqrt{10}} \end{pmatrix}, \qquad \mathbf{u}_2 = \frac{\mathbf{w}_2}{\|\mathbf{w}_2\|} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \qquad \mathbf{u}_3 = \frac{\mathbf{w}_3}{\|\mathbf{w}_3\|} = \begin{pmatrix} \frac{1}{\sqrt{10}} \\ 0 \\ \frac{3}{\sqrt{10}} \end{pmatrix}.$$

Combine $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ into one matrix to construct the orthogonal matrix $Q = \begin{pmatrix} \frac{3}{\sqrt{10}} & 0 & \frac{1}{\sqrt{10}} \\ 0 & 1 & 0 \\ -\frac{1}{\sqrt{10}} & 0 & \frac{3}{\sqrt{10}} \end{pmatrix}$. Then the upper triangular matrix

$$R = Q^T A = \begin{pmatrix} \frac{3}{\sqrt{10}} & 0 & \frac{1}{\sqrt{10}} \\ 0 & 1 & 0 \\ -\frac{1}{\sqrt{10}} & 0 & \frac{3}{\sqrt{10}} \end{pmatrix} \begin{pmatrix} 3 & 0 & 2 \\ 0 & 4 & 1 \\ -1 & 0 & 1 \end{pmatrix} \begin{pmatrix} \sqrt{10} & 0 & \frac{5}{\sqrt{10}} \\ 0 & 4 & 1 \\ 0 & 0 & \frac{5}{\sqrt{10}} \end{pmatrix}.$$

□

5. (18 points) (a) Describe all least-squares solutions of the system

$$\begin{aligned} x + 2y &= 2 \\ x + 2y &= 3 \\ x + 2y &= 4 \end{aligned}$$

SOLUTION. The coefficient matrix $A = \begin{pmatrix} 1 & 2 \\ 1 & 2 \\ 1 & 2 \end{pmatrix}$, the right hand sides $\mathbf{b} = \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix}$. The least-squares solutions of the system $A\mathbf{x} = \mathbf{b}$ are solutions to the system $A^T A\mathbf{x} = A^T\mathbf{b}$. Apply Gaussian,

$$A^T(A|\mathbf{b}) = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \end{pmatrix} \left(\begin{array}{cc|c} 1 & 2 & 2 \\ 1 & 2 & 3 \\ 1 & 2 & 4 \end{array} \right) = \left(\begin{array}{cc|c} 3 & 6 & 9 \\ 6 & 12 & 18 \end{array} \right) \xrightarrow{R_2 - 2R_1} \left(\begin{array}{cc|c} 3 & 6 & 9 \\ 0 & 0 & 0 \end{array} \right).$$

The general solution is

$$\hat{\mathbf{x}} = \begin{pmatrix} \hat{x} \\ \hat{y} \end{pmatrix} = \begin{pmatrix} 3 - 2\hat{y} \\ \hat{y} \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \end{pmatrix} + \hat{y} \begin{pmatrix} -2 \\ 1 \end{pmatrix},$$

which form all least-squares solutions to the system $A\mathbf{x} = \mathbf{b}$.

(b) Let $\mathbf{b} = (2, 3, 4)^T$. Find the vector $\mathbf{v} \in \operatorname{span}\{(1, 1, 1)^T\}$ such that $\|\mathbf{v} - \mathbf{b}\|$ is minimized.

SOLUTION. Let $\mathbf{w} = (1, 1, 1)^T$. The closest point is the orthogonal projection of $\mathbf{b}$ onto the subspace $W = \operatorname{span}\{\mathbf{w}\}$. By the orthogonal projection formula,

$$\operatorname{proj}_W \mathbf{b} = \frac{\langle \mathbf{b}, \mathbf{w} \rangle}{\|\mathbf{w}\|^2} \mathbf{w} = \frac{9}{3} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \\ 3 \end{pmatrix}.$$

□

6. (15 points) Prove that if $\|Q\mathbf{x}\| = \|\mathbf{x}\|$ for all $\mathbf{x} \in \mathbb{R}^n$, then Q is an orthogonal matrix.

PROOF. Since $\|Q\mathbf{x}\| = \|\mathbf{x}\|$, we know that for all $\mathbf{x} \in \mathbb{R}^n$,

$$p(\mathbf{x}) \doteq \sum_{i=1}^{n} x_i^2 = \|\mathbf{x}\|^2 = \|Q\mathbf{x}\|^2 = (Q\mathbf{x})^T (Q\mathbf{x}) = \mathbf{x}^T Q^T Q \mathbf{x}.$$

Now $p(\mathbf{x})$ is a quadratic form, its associated symmetric matrix K is the identity matrix. (Why?) Since $Q^T Q$ is also symmetric, we can conclude $Q^T Q = K = I$, which implies that Q is an orthogonal matrix. □

Bonus. A square matrix A satisfies $A^T = -A$. Show that

(a) (5 points) $I - A$ is always invertible. (2 points for demonstration by an example)

PROOF. Suppose $\mathbf{x}$ satisfies $(I - A)\mathbf{x} = \mathbf{0}$. Then $A\mathbf{x} = I\mathbf{x} = \mathbf{x}$. Thus,

$$\mathbf{x}^T\mathbf{x} = (A\mathbf{x})^T \mathbf{x} = \mathbf{x}^T A^T \mathbf{x} = \mathbf{x}^T (-A)\mathbf{x} = -\mathbf{x}^T (A\mathbf{x}) = -\mathbf{x}^T \mathbf{x}.$$

This shows $\|\mathbf{x}\|^2 = \mathbf{x}^T\mathbf{x} = 0$, namely $\mathbf{x} = 0$. Therefore, the homogeneous system $(I - A)\mathbf{x} = \mathbf{0}$ has only the trivial solution. So $I - A$ is invertible.

(b) (5 points) $Q = (I - A)^{-1}(I + A)$ is an orthogonal matrix. (2 points for demonstration by an example)

PROOF. We are to show $QQ^T = I$.

$$\begin{aligned}
QQ^T &= (I - A)^{-1}(I + A)((I - A)^{-1}(I + A))^T \\
&= (I - A)^{-1}(I + A)(I + A)^T((I - A)^{-1})^T \\
&= (I - A)^{-1}(I + A)(I + A^T)((I - A)^T)^{-1} \\
&= (I - A)^{-1}(I + A)(I - A)(I - A^T)^{-1} \\
&= (I - A)^{-1}(I - A^2)(I + A)^{-1} \\
&= (I - A)^{-1}(I - A)(I + A)(I + A)^{-1} \\
&= I.
\end{aligned}$$

The third and the second equations from the last are the key steps, showing that $I + A$ and $I - A$ are commutative under matrix multiplication. Therefore Q is an orthogonal matrix. □