21-241: Matrix Algebra – Summer I, 2006

Exam 2 Solutions

1. (18 points) True or False. (Don't need to justify)

   (a) The set of all vectors of the form $\begin{pmatrix} 3a+b \\ 4 \\ a-5b \end{pmatrix}$, where a, b represent arbitrary real numbers, is a vector space.

   SOLUTION. **False**. The zero vector $\mathbf{0}$ doesn't belong to the set.

   (b) $\mathbb{R}^2$ is a subspace of $\mathbb{R}^3$.

   SOLUTION. **False**. $\mathbb{R}^2$ is not a subset of $\mathbb{R}^3$.

   (c) If U is the echelon form of A, then $\operatorname{Col} U = \operatorname{Col} A$.

   SOLUTION. **False**. A simple counter-example: $A = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $U = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

   (d) The kernel of an $m \times n$ matrix is in $\mathbb{R}^m$.

   SOLUTION. **False**. The kernel of an $m \times n$ matrix is in $\mathbb{R}^n$, while the range is in $\mathbb{R}^m$.

   (e) A positive definite quadratic form q satisfies $q(\mathbf{x}) > 0$ for all $\mathbf{x}$ in $\mathbb{R}^n$.

   SOLUTION. **False**. $q(\mathbf{0}) = 0$.

   (f) If A is a square matrix, $\mathbf{u} \in \operatorname{Col} A$, $\mathbf{v} \in \ker A$, then $\mathbf{u} \perp \mathbf{v}$.

   SOLUTION. **False**. For example, $A = \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}$, $\mathbf{u} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \in \operatorname{Col} A$, $\mathbf{v} = \begin{pmatrix} 2 \\ 2 \end{pmatrix} \in \ker A$. But $\mathbf{u} \cdot \mathbf{v} \neq 0$, so, $\mathbf{u} \not\perp \mathbf{v}$. $\square$

2. (15 points) Find the value(s) of h for which the vectors $\begin{pmatrix} 1 \\ 5 \\ -3 \end{pmatrix}$, $\begin{pmatrix} -2 \\ -9 \\ 6 \end{pmatrix}$, $\begin{pmatrix} 3 \\ h \\ -9 \end{pmatrix}$ are linearly *dependent*.

   SOLUTION. Let matrix A have the three vectors as its columns. Apply Gaussian to reduce A in the echelon form:

$$\begin{pmatrix} 1 & -2 & 3 \\ 5 & -9 & h \\ -3 & 6 & -9 \end{pmatrix} \xrightarrow[R_3+3R_1]{R_2-5R_1} \begin{pmatrix} \boxed{1} & -2 & 3 \\ 0 & \boxed{1} & h-15 \\ 0 & 0 & 0 \end{pmatrix}$$

   We see that A always has only 2 pivots regardless of the value of h. Therefore, the three vectors are linearly dependent for any real number h. $\square$

3. (18 points)

   (a) Prove that the only element $\mathbf{w}$ in an inner product space V that is orthogonal to every vector is the zero vector $\mathbf{w} = \mathbf{0}$.

   PROOF. Since $\mathbf{w}$ is orthogonal to every vector, it's orthogonal to itself, namely $\langle \mathbf{w}, \mathbf{w} \rangle = 0$. By positivity, $\mathbf{w} = 0$. $\square$

(b) Prove that $\|\mathbf{w}\| \leqslant \|\mathbf{v}\| + \|\mathbf{v}+\mathbf{w}\|$ for any $\mathbf{v}, \mathbf{w} \in V$.

PROOF. By the triangle inequality,

$$\|\mathbf{w}\| = \|(\mathbf{v}+\mathbf{w}) + (-\mathbf{v})\| \leqslant \|\mathbf{v}+\mathbf{w}\| + \|-\mathbf{v}\| = \|\mathbf{v}\| + \|\mathbf{v}+\mathbf{w}\|.$$

$\square$

4. (16 points) Show that $\mathbf{v}_1 = \begin{pmatrix} 4 \\ 1 \\ 3 \end{pmatrix}$, $\mathbf{v}_2 = \begin{pmatrix} 2 \\ -2 \\ -1 \end{pmatrix}$ and $\mathbf{w}_1 = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$, $\mathbf{w}_2 = \begin{pmatrix} -6 \\ 2 \\ -1 \end{pmatrix}$ are two bases for the same two-dimensional subspace $V \in \mathbb{R}^3$.

PROOF. Let $A = (\mathbf{v}_1\ \mathbf{v}_2)$, $B = (\mathbf{w}_1\ \mathbf{w}_2)$. We are to characterize $\operatorname{Col} A$ and $\operatorname{Col} B$. First consider the system $A\mathbf{x} = \mathbf{b}$, where the right hand side will remain unspecified for the moment. Apply Gaussian to the augmented matrix:

$$\left(\begin{array}{cc|c} 4 & 2 & b_1 \\ 1 & -2 & b_2 \\ 3 & -1 & b_3 \end{array}\right) \xrightarrow{R_1 \leftrightarrow R_2} \left(\begin{array}{cc|c} 1 & -2 & b_2 \\ 4 & 2 & b_1 \\ 3 & -1 & b_3 \end{array}\right) \xrightarrow[R_3-3R_1]{R_2-4R_1} \left(\begin{array}{cc|c} 1 & -2 & b_2 \\ 0 & 10 & b_1-4b_2 \\ 0 & 5 & b_3-3b_2 \end{array}\right)$$

$$\xrightarrow{R_3-\frac{1}{2}R_2} \left(\begin{array}{cc|c} \boxed{1} & -2 & b_2 \\ 0 & \boxed{10} & b_1-4b_2 \\ 0 & 0 & -\frac{1}{2}b_1 - b_2 + b_3 \end{array}\right)$$

Hence, $\operatorname{Col} A = \{(b_1, b_2, b_3)^T \mid -\frac{1}{2}b_1 - b_2 + b_3 = 0\}$, a two-dimensional subspace of $\mathbb{R}^3$. Similarly,

$$\left(\begin{array}{cc|c} 2 & -6 & b_1 \\ 0 & 2 & b_2 \\ 1 & -1 & b_3 \end{array}\right) \xrightarrow{R_1 \leftrightarrow R_3} \left(\begin{array}{cc|c} 1 & -1 & b_3 \\ 0 & 2 & b_2 \\ 2 & -6 & b_1 \end{array}\right) \xrightarrow{R_3-2R_1} \left(\begin{array}{cc|c} 1 & -1 & b_3 \\ 0 & 2 & b_2 \\ 0 & -4 & b_1-2b_3 \end{array}\right)$$

$$\xrightarrow{R_3+2R_2} \left(\begin{array}{cc|c} \boxed{1} & -1 & b_3 \\ 0 & \boxed{2} & b_2 \\ 0 & 0 & b_1+2b_2-2b_3 \end{array}\right)$$

So, $\operatorname{Col} B = \{(b_1, b_2, b_3)^T \mid b_1 + 2b_2 - 2b_3 = 0\}$. Clearly, $\operatorname{Col} A = \operatorname{Col} B$, denoted by V. Then $\{\mathbf{v}_1, \mathbf{v}_2\}$ and $\{\mathbf{w}_1, \mathbf{w}_2\}$ are two bases for V, which is a two-dimensional subspace of $\mathbb{R}^3$. $\square$

5. (15 points) Show that the hyperplane $x + 2y + z - w = 0$ is a subspace of $\mathbb{R}^4$. Find the dimension of and a basis for the hyperplane.

SOLUTION. Denote the hyperplane by H. Any vector in H is actually a solution to the homogeneous system

$$x + 2y + z - w = 0.$$

So we are to find a basis for the kernel of the coefficient matrix $A = \begin{pmatrix} 1 & 2 & 1 & -1 \end{pmatrix}$, which is already in the echelon form. Clearly, y, z, w are free variables, and $x = -2y - z + w$. So the general solution can be written as

$$\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} -2y-z+w \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} -2y \\ y \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} -z \\ 0 \\ z \\ 0 \end{pmatrix} + \begin{pmatrix} w \\ 0 \\ 0 \\ w \end{pmatrix} = y\begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + z\begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix} + w\begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

Therefore, $H = \operatorname{span}\{(-2,1,0,0)^T, (-1,0,1,0)^T, (1,0,0,1)^T\}$. Since any span is a subspace, H is a subspace of $\mathbb{R}^4$. Moreover, the three vectors spanning H form a basis for H, and thus $\dim H = 3$. $\square$

6. (18 points) Write the quadratic form $q(\mathbf{x}) = 3x_1^2 + 5x_2^2 + 6x_3^2 + 4x_4^2 + 4x_1x_2 - 2x_1x_3 - 2x_2x_4 - x_3x_4$ in the form $q(\mathbf{x}) = \mathbf{x}^T K \mathbf{x}$ for some symmetric matrix K. Determine if $q(\mathbf{x})$ is positive definite.

SOLUTION. Matrix K can be directly written out from the coefficients of the quadratic form q:

$$K = \begin{pmatrix} 3 & 2 & -1 & 0 \\ 2 & 5 & 0 & -1 \\ -1 & 0 & 6 & -\frac{1}{2} \\ 0 & -1 & -\frac{1}{2} & 4 \end{pmatrix}$$

We apply Gaussian to K:

$$\begin{pmatrix} 3 & 2 & -1 & 0 \\ 2 & 5 & 0 & -1 \\ -1 & 0 & 6 & -\frac{1}{2} \\ 0 & -1 & -\frac{1}{2} & 4 \end{pmatrix} \xrightarrow[R_3+\frac{1}{3}R_1]{R_2-\frac{2}{3}R_1} \begin{pmatrix} 3 & 2 & -1 & 0 \\ 0 & \frac{11}{3} & \frac{2}{3} & -1 \\ 0 & \frac{2}{3} & \frac{20}{3} & -\frac{1}{2} \\ 0 & -1 & -\frac{1}{2} & 4 \end{pmatrix} \xrightarrow[R_4+\frac{3}{11}R_2]{R_3-\frac{2}{11}R_2} \begin{pmatrix} 3 & 2 & -1 & 0 \\ 0 & \frac{11}{3} & \frac{2}{3} & -1 \\ 0 & 0 & \frac{72}{11} & -\frac{7}{22} \\ 0 & 0 & -\frac{7}{22} & \frac{41}{11} \end{pmatrix}$$

$$\xrightarrow{R_4+\frac{7}{144}R_3} \begin{pmatrix} \boxed{3} & 2 & -1 & 0 \\ 0 & \boxed{\frac{11}{3}} & \frac{2}{3} & -1 \\ 0 & 0 & \boxed{\frac{72}{11}} & -\frac{7}{22} \\ 0 & 0 & 0 & \boxed{\frac{1069}{288}} \end{pmatrix}$$

Hence, K is regular and has all positive pivots. Thus K is positive definite, so is $q(\mathbf{x})$. $\square$

Bonus. *This problem may be much more challenging than the others. Allocate your time wisely.*

(7 points) Prove that $(a+2b+3c)^2 \leqslant 6\,(a^2+2b^2+3c^2)$ for any real numbers a, b, c.

PROOF. Let $\mathbf{w}_1 = (1,1,1)^T$, $\mathbf{w}_2 = (a,b,c)^T$. Using weighted inner product $\langle \mathbf{u}, \mathbf{v} \rangle = u_1v_1 + 2u_2v_2 + 3u_3v_3$ on $\mathbb{R}^3$, we get

$$\langle \mathbf{w}_1, \mathbf{w}_2 \rangle = a + 2b + 3c, \qquad \|\mathbf{w}_1\|^2 = 6, \qquad \|\mathbf{w}_2\|^2 = a^2 + 2b^2 + 3c^2.$$

Apply Cauchy-Schwarz inequality, we obtain:

$$(a+2b+3c)^2 \leqslant 6\,(a^2+2b^2+3c^2).$$

(3 points) Prove the same inequality in a different way.

PROOF. We can also prove the inequality by completing squares. Since

$$\begin{aligned} & 6\,(a^2+2b^2+3c^2) - (a+2b+3c)^2 \\ =\ & 5a^2 + 8b^2 + 9c^2 - 4ab - 12bc - 6ca \\ =\ & 2(a-b)^2 + 6(b-c)^2 + 3(c-a)^2 \geqslant 0, \end{aligned}$$

we may obtain $(a+2b+3c)^2 \leqslant 6\,(a^2+2b^2+3c^2)$. Moreover, it's clear that the equality holds if only and only if $a = b = c$. $\square$