21-241: Matrix Algebra – Summer I, 2006

Exam 1 Solutions

1. (18 points) True or False. (Don't need to justify)

   (a) If A and B are 3×3 and $B = (\mathbf{b}_1\ \mathbf{b}_2\ \mathbf{b}_3)$, then $AB = (A\mathbf{b}_1 + A\mathbf{b}_2 + A\mathbf{b}_3)$.
   SOLUTION. **False**. $AB = (A\mathbf{b}_1\ A\mathbf{b}_2\ A\mathbf{b}_3)$.

   (b) If A is invertible, then elementary row operations that reduce A to the identity I_n also reduce A^{-1} to I_n.
   SOLUTION. **False**. The elementary row operations that reduce A to the identity amount to left multiplying a matrix by A^{-1}, but $A^{-1} \cdot A^{-1} = A^{-2} \neq I$ in general.

   (c) One can find an $m \times n$ matrix of rank r for any $0 \leqslant r \leqslant \min\{m, n\}$.
   SOLUTION. **True**. Zero matrix has rank 0. For any $1 \leqslant r \leqslant \min\{m, n\}$, let the first r diagonal entries be 1, all other entries be 0. Then this matrix has rank r.

   (d) Every permutation matrix satisfies $P^2 = I$.
   SOLUTION. **False**. Only elementary permutation matrices satisfy $P^2 = I$.

   (e) If A has a zero entry on its diagonal, it is not regular.
   SOLUTION. **False**. Let $A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$, for example. Then A has a zero on its diagonal, but it's still regular.

   (f) If A and B are $n \times n$ matrices and B is invertible, then $\det(AB^{-1}) = \dfrac{\det A}{\det B}$.
   SOLUTION. **True**. $\det(AB^{-1}) = (\det A)(\det B^{-1}) = \dfrac{\det A}{\det B}$, since $\det B^{-1} = \dfrac{1}{\det B}$. □

2. (15 points) Apply Gaussian elimination (not Gauss-Jordan) to the linear system

$$\begin{aligned} x_2 - 2x_3 + 2x_4 &= 2 \\ 2x_1 + 3x_2 + 6x_3 + 3x_4 &= 11 \\ -7x_2 + 14x_3 + 7x_4 &= 7 \\ -x_2 + 2x_3 + x_4 &= 1 \end{aligned}$$

Do no more work than necessary to determine whether the system has (i) no solution, (ii) infinitely many solutions, or (iii) a unique solution. Then indicate which is the case, and how you can tell.

SOLUTION. Note that scaling is not permitted in Gaussian elimination. We reduce the augmented matrix to echelon form by only replacement and interchange operations.

$$\left(\begin{array}{cccc|c} 0 & 1 & -2 & 2 & 2 \\ 2 & 3 & 6 & 3 & 11 \\ 0 & -7 & 14 & 7 & 7 \\ 0 & -1 & 2 & 1 & 1 \end{array}\right) \xrightarrow{R_1 \leftrightarrow R_2} \left(\begin{array}{cccc|c} 2 & 3 & 6 & 3 & 11 \\ 0 & 1 & -2 & 2 & 2 \\ 0 & -7 & 14 & 7 & 7 \\ 0 & -1 & 2 & 1 & 1 \end{array}\right) \xrightarrow[R_4+R_2]{R_3+7R_2} \left(\begin{array}{cccc|c} 2 & 3 & 6 & 3 & 11 \\ 0 & 1 & -2 & 2 & 2 \\ 0 & 0 & 0 & 21 & 21 \\ 0 & 0 & 0 & 3 & 3 \end{array}\right)$$

$$\xrightarrow{R_4-(1/7)R_3} \left(\begin{array}{cccc|c} 2 & 3 & 6 & 3 & 11 \\ 0 & 1 & -2 & 2 & 2 \\ 0 & 0 & 0 & 21 & 21 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

Now we can see the system is consistent and has one free variable (x_3). So there are infinitely many solution. □

3. (20 points) (a) Show that if A has size $n \times n$, then $\det(-A) = (-1)^n \det A$.

PROOF. Note that $-A$ amounts to multiplying each row of A by -1. Since multiplying a row by a scalar multiplies the determinant by the same scalar, and there are totally n rows, we have $\det(-A) = (-1)^n \det A$. In general, $\det(cA) = c^n \det A$. $\square$

(b) Find the determinant of the matrix $\begin{pmatrix} 1 & 5 & 9 & 13 \\ 2 & 6 & 10 & 14 \\ 3 & 7 & 11 & 15 \\ 4 & 8 & 12 & 16 \end{pmatrix}$.

SOLUTION. By doing some replacement operations that don't affect the determinant, we get

$$\begin{pmatrix} 1 & 5 & 9 & 13 \\ 2 & 6 & 10 & 14 \\ 3 & 7 & 11 & 15 \\ 4 & 8 & 12 & 16 \end{pmatrix} \xrightarrow[R_4-R_3]{R_2-R_1} \begin{pmatrix} 1 & 5 & 9 & 13 \\ 1 & 1 & 1 & 1 \\ 3 & 7 & 11 & 15 \\ 1 & 1 & 1 & 1 \end{pmatrix} \xrightarrow{R_4-R_2} \begin{pmatrix} 1 & 5 & 9 & 13 \\ 1 & 1 & 1 & 1 \\ 3 & 7 & 11 & 15 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

which contains a row of all zeros. Therefore, the determinant of the matrix equals 0. $\square$

4. (15 points) Let $A = \begin{pmatrix} -2 & -7 & -9 \\ 2 & 5 & 6 \\ 1 & 3 & 4 \end{pmatrix}$. Find the third column of A^{-1} ***without*** computing the other columns.

SOLUTION. To compute the third column of A^{-1}, we only need to solve the system $A\mathbf{x} = (0\ 0\ 1)^T$, where the right hand side is the third column of identity I_3. Using Gaussian elimination, we get

$$\left(\begin{array}{ccc|c} -2 & -7 & -9 & 0 \\ 2 & 5 & 6 & 0 \\ 1 & 3 & 4 & 1 \end{array}\right) \xrightarrow{R_1 \leftrightarrow R_3} \left(\begin{array}{ccc|c} 1 & 3 & 4 & 1 \\ 2 & 5 & 6 & 0 \\ -2 & -7 & -9 & 0 \end{array}\right) \xrightarrow[R_3+2R_1]{R_2-2R_1} \left(\begin{array}{ccc|c} 1 & 3 & 4 & 1 \\ 0 & -1 & -2 & -2 \\ 0 & -1 & -1 & 2 \end{array}\right)$$

$$\xrightarrow{R_3-R_2} \left(\begin{array}{ccc|c} 1 & 3 & 4 & 1 \\ 0 & -1 & -2 & -2 \\ 0 & 0 & 1 & 4 \end{array}\right)$$

Then applying back substitution, we have $x_3 = 4$, $x_2 = -6$, $x_1 = 3$. Therefore, the third column of A^{-1} is $(3\ -6\ 4)^T$. $\square$

5. (16 points) (a) Suppose A is a 3×3 matrix with three pivots. Does the equation $A\mathbf{x} = 0$ have a nontrivial solution? Does the equation $A\mathbf{x} = \mathbf{b}$ have at least one solution for every possible $\mathbf{b}$? Explain.

SOLUTION. Since A is a 3×3 matrix with three pivots, we know

- Each row has a pivot. So the system $A\mathbf{x} = \mathbf{b}$ is consistent, i.e., has at least one solution, for every possible $\mathbf{b}$.
- Each column has a pivot. So there is no free variable, the system can't have infinitely many solutions.

Therefore, the system always has a unique solution. In homogeneous case, this unique solution is the trivial one. So $A\mathbf{x} = 0$ has no nontrivial solution, $A\mathbf{x} = \mathbf{b}$ has at least one solution. $\square$

(b) What if A is a 2×4 matrix with two pivots instead?

SOLUTION. If A is a 2×4 matrix with two pivots, we now know that

- Each row has a pivot. So the system $A\mathbf{x} = \mathbf{b}$ is consistent, i.e., has at least one solution, for every possible $\mathbf{b}$.
- Two rows have no pivot. So there are two free variables, the system has infinitely many solutions.

Therefore, $A\mathbf{x} = 0$ has (infinitely many) nontrivial solutions, $A\mathbf{x} = \mathbf{b}$ has at least one solution. □

6. (16 points) Find a permuted LU factorization of the following complex matrix:

$$\begin{pmatrix} 0 & 1 & -\mathrm{i} \\ \mathrm{i} & 0 & -1 \\ -1 & \mathrm{i} & 1 \end{pmatrix}$$

SOLUTION. The factorization is not unique. The following is a possible one.

$$\begin{pmatrix} 0 & 1 & -\mathrm{i} \\ \mathrm{i} & 0 & -1 \\ -1 & \mathrm{i} & 1 \end{pmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{pmatrix} \mathrm{i} & 0 & -1 \\ 0 & 1 & -\mathrm{i} \\ -1 & \mathrm{i} & 1 \end{pmatrix} \xrightarrow[R_3 - \mathrm{i}R_2]{R_3 - \mathrm{i}R_1} \begin{pmatrix} \mathrm{i} & 0 & -1 \\ 0 & 1 & -\mathrm{i} \\ 0 & 0 & \mathrm{i} \end{pmatrix} = U$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow[R_3 + \mathrm{i}R_2]{R_3 + \mathrm{i}R_1} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \mathrm{i} & \mathrm{i} & 1 \end{pmatrix} = L$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = P$$

So, $\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & -\mathrm{i} \\ \mathrm{i} & 0 & -1 \\ -1 & \mathrm{i} & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \mathrm{i} & \mathrm{i} & 1 \end{pmatrix} \begin{pmatrix} \mathrm{i} & 0 & -1 \\ 0 & 1 & -\mathrm{i} \\ 0 & 0 & \mathrm{i} \end{pmatrix}.$ □

Bonus. (10 points) Let $\mathbf{v}$, $\mathbf{w}$ be $n \times 1$ column vectors. Prove that when $\mathbf{w}^T\mathbf{v} \neq 1$, there exists a scalar c such that the inverse of the $n \times n$ matrix $A = I - \mathbf{v}\mathbf{w}^T$ has the form $A^{-1} = I - c\mathbf{v}\mathbf{w}^T$.

PROOF. Suppose $A^{-1} = I - c\mathbf{v}\mathbf{w}^T$. We are to determine the scalar c. Since

$$\begin{aligned} I = A^{-1}A &= (I - c\mathbf{v}\mathbf{w}^T)(I - \mathbf{v}\mathbf{w}^T) \\ &= I - c\mathbf{v}\mathbf{w}^T - \mathbf{v}\mathbf{w}^T + (c\mathbf{v}\mathbf{w}^T)(\mathbf{v}\mathbf{w}^T) && \text{(by distributivity)} \\ &= I - c\mathbf{v}\mathbf{w}^T - \mathbf{v}\mathbf{w}^T + c\mathbf{v}(\mathbf{w}^T\mathbf{v})\mathbf{w}^T && \text{(by associativity)} \\ &= I - c\mathbf{v}\mathbf{w}^T - \mathbf{v}\mathbf{w}^T + c(\mathbf{w}^T\mathbf{v})\mathbf{v}\mathbf{w}^T && \text{(since } \mathbf{w}^T\mathbf{v} \text{ is a scalar)} \\ &= I - (c - 1 - c\mathbf{w}^T\mathbf{v})\mathbf{v}\mathbf{w}^T, \end{aligned}$$

we only need to choose c such that $c - 1 - c\mathbf{w}^T\mathbf{v} = 0$, or $c = 1/(1 - \mathbf{w}^T\mathbf{v})$. This is doable because $\mathbf{w}^T\mathbf{v} \neq 1$ implies the denominator doesn't equal 0. Thus we complete the proof. □