Snelson Tensegrity Structures

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Summary

Tensegrity structures were first constructed by the artist Kenneth Snelson in 1948. This type of structure was named and popularized by Buckminster Fuller as possible architectural constructs. Since the mid-seventies, generalized versions of these constructs, namely structures some of whose elements cannot sustain compression, have been studied by both engineers and mathematicians. With I. J. Oppenheim [1, 2, 3], the author has begun a study of the structures which are of the type built by Snelson. Here we present some results which help to characterize the equilibria of such constructs, and discuss a technique for describing the evolution of the positions.

1 Introduction

Following sculptures created by Snelson in 1948, in 1961 Buckminster Fuller patented a class of cable-bar structures which he called tensegrity structures [4, 5]. These consisted of arrangements with bars in compression, no two connected directly, with structural integrity maintained by the tension in the cables. Hence "tension-integrity", compressed to "tensegrity". These structures, remarkable to Fuller for enclosing large volumes of space with minimal weight, are not as well known as his corresponding shell constructions, but offer interest both

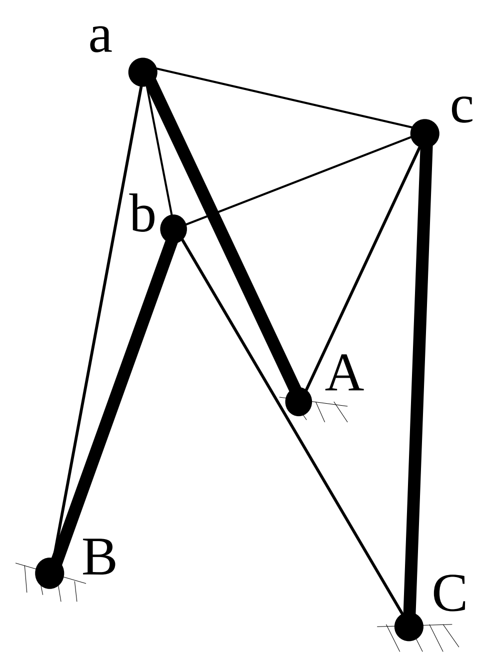

Figure 1: T-3 Structure (Not in Equilibrium)

mathematically and as minimal weight engineering constructions. (See [6] and [7].) The Snelson-style constructions still are the most useful as examples, since they are minimalist, in a sense which we make precise below. The simplest three-dimensional version of such a structure is the base-anchored T-3 (following a classification of [8]) shown in Figure 1.

2 Geometry & Mechanics

Physically, a tensegrity structure is a pin-connected truss in which elements may be either bars or cables. Mathematically, we can describe the structure as a bi-graph $\mathcal{S}$ together with a specification of a set of edge-lengths. The graph consists in a set of **edges** $\mathcal{E}$ and a set of **nodes** $\mathcal{N}$. The edges are divided into two sets, of **bars** and **cables**:

$$\mathcal{E} = \mathcal{B} \cup \mathcal{C}. \tag{1}$$

If $e \in \mathcal{E}$ is an edge we may find it convenient to denote its end-nodes generically as

$$e_\alpha, e_\omega \in \mathcal{N}. \tag{2}$$

To complete the prescription of the structure we assign a set of **edge-lengths**

$$\mathbf{\Lambda} \in \mathbb{R}^{\mathcal{E}}, \tag{3}$$

representing the squares of the lengths of all edges.

A **placement** of the graph $\mathcal{S}$ is an assignment of nodes to points in $\mathbb{R}^3$. It is most convenient to describe the placement as a vector in a product space:

$$\vec{\boldsymbol{p}} \in \left(\mathbb{R}^3\right)^{\mathcal{N}}. \tag{4}$$

Soon we will confine ourselves to the special case in which the structures are pinned to ground, which means that, as for our example, three non-collinear nodes are restricted to fixed positions.

Placements will be required to respect prescribed edge-lengths. We quantify this by introducing the (squared) **length-map**

$$\begin{aligned} \boldsymbol{\lambda} &: \left(\mathbb{R}^3\right)^{\mathcal{N}} \to \mathbb{R}^{\mathcal{E}} \\ (\boldsymbol{\lambda}(\vec{\boldsymbol{p}}))_e &= \frac{1}{2}||\vec{\boldsymbol{p}}_{e_\alpha} - \vec{\boldsymbol{p}}_{e_\omega}||^2. \end{aligned} \tag{5}$$

Given $\mathbf{\Lambda}$, a placement $\vec{\boldsymbol{p}}$ is **admissible** if

$$\boldsymbol{\lambda}_b(\vec{\boldsymbol{p}}) = \mathbf{\Lambda}_b, \ \forall b \in \mathcal{B} \text{ and} \tag{6a}$$

$$\boldsymbol{\lambda}_c(\vec{\boldsymbol{p}}) \leq \mathbf{\Lambda}_c, \ \forall c \in \mathcal{C}. \tag{6b}$$

We abbreviate this as

$$\boldsymbol{\lambda} \leq \mathbf{\Lambda}. \tag{7}$$

A **motion** of $\mathcal{S}$ is a piece-wise-smooth one-parameter family of placements. The derivative of a motion is an instance of an **velocity**

$$\vec{\boldsymbol{v}} \in \left(\mathbb{R}^3\right)^{\mathcal{N}}, \tag{8}$$

which assigns a velocity vector in $\mathbb{R}^3$ to each node of the structure. When we restrict ourselves to the case in which three nodes are pinned to ground, some of these vectors will be restricted to be zero.

If $\vec{\boldsymbol{q}}(t)$ is a motion initiating at $\vec{\boldsymbol{p}}$, then the rate of change of the length function at time 0 is given by

$$\frac{d}{dt}\lambda_e(\vec{\boldsymbol{q}})(0) = (\vec{\boldsymbol{p}}_{e_\alpha} - \vec{\boldsymbol{p}}_{e_\omega}) \cdot (\dot{\vec{\boldsymbol{q}}}_{e_\alpha} - \dot{\vec{\boldsymbol{q}}}_{e_\omega}). \tag{9}$$

Generalizing this, we assign to any placement and any velocity the (rate-of) stretching vector

$$\left(\vec{\boldsymbol{p}}_{e_\alpha} - \vec{\boldsymbol{p}}_{e_\omega}\right) \cdot (\vec{\boldsymbol{v}}_{e_\alpha} - \vec{\boldsymbol{v}}_{e_\omega}). \tag{10}$$

This represents the rate of lengthening of the edge e times the length of the edge. This relation is simplified to

$$\boldsymbol{\pi}_e(\vec{\boldsymbol{p}}) \cdot \vec{\boldsymbol{v}} \tag{11}$$

when we introduce the **edge vector**

$$\boldsymbol{\pi}_e(\vec{\boldsymbol{p}}) = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \vec{\boldsymbol{p}}_{e_\alpha} - \vec{\boldsymbol{p}}_{e_\omega} \\ 0 \\ \cdot \\ 0 \\ \vec{\boldsymbol{p}}_{e_\omega} - \vec{\boldsymbol{p}}_{e_\alpha} \\ 0 \\ \vdots \\ 0 \end{bmatrix} \in \left(\mathbb{R}^3\right)^{\mathcal{N}}. \quad (12)$$

Here, as suggested by the notation, the non-zero entries occur in the e_α and e_ω slots, respectively.

Each edge vector is a linear function of $\vec{\boldsymbol{p}}$ and we may write

$$\boldsymbol{\pi}_e = B_e \vec{\boldsymbol{p}} \quad (13)$$

for each e in $\mathcal{E}$. The transformations B_e are symmetric, positive, and

$$B_e^2 = 2B_e \quad (14)$$

A final convenient notation: we construct the **geometrical matrix**, also called the rigidity matrix, $\Pi : \mathbb{R}^{\mathcal{E}} \to (\mathbb{R}^3)^{\mathcal{N}}$, by using the edge vectors as columns. Then the totality of the relations (11) is given as $\Pi^\top \vec{\boldsymbol{v}}$.

For the remainder of the paper, we shall simplify the kinematics by restricting placements in a way which eliminates the possibility of rigid-body motion. In particular, for T-3, with nodal labels as in Figure 1, $\vec{\boldsymbol{p}}_A$, $\vec{\boldsymbol{p}}_B$, and $\vec{\boldsymbol{p}}_C$ are taken to be fixed in non-collinear positions. We can then formally truncate the nodal set $\mathcal{N}$ to $\{a, b, c\}$, and correspondingly truncate the placement and velocity space, reducing Π to nine columns, each of dimension nine. (Of course $\vec{\boldsymbol{p}}_A$, $\vec{\boldsymbol{p}}_B$, and $\vec{\boldsymbol{p}}_C$ still appear in Π.) See [9] for a discussion of the unrestricted case.

3 Rigidity & Stability

There may be motions $\vec{\boldsymbol{q}}(t)$ away from $\vec{\boldsymbol{p}}$ which are compatible with the length restriction

$$\boldsymbol{\lambda}(\vec{\boldsymbol{q}}(t)) \leq \boldsymbol{\lambda}(\vec{\boldsymbol{p}}). \quad (15)$$

We would call such a $\vec{\boldsymbol{q}}$ an **admissible motion**. The velocities of admissible motions belong to the (closed, convex) cone of admissible velocities, or **flexures**:

$$\left\{ \vec{\boldsymbol{v}} \,|\, \Pi^\top \vec{\boldsymbol{v}} \leq \mathbf{0} \right\}, \quad (16)$$

where $\leq$ has the same interpretation as in (7).

Finally, we define the fundamental kinematic notions of the theory:

Definition 3.1 *A placement* $\vec{\boldsymbol{p}}$ *is* **rigid** *if it admits no flexures.*

Definition 3.2 *A placement* $\vec{\boldsymbol{p}}$ *is* **stable** *if it admits no admissible motions.*

It is clear that any rigid placement is stable; the converse fails in general. A condition which ensures that the converse implication hold is maximal independence of collections of edge vectors [10]:

Theorem 3.3 (Roth & Whiteley) *If* $\vec{\boldsymbol{p}}$ *is such that for each* $\mathcal{A} \subset \mathcal{E}$ *the set of edge vectors* $\{\boldsymbol{\pi}_e \,|\, e \in \mathcal{A}\}$ *has span the maximal possible dimension (amongst all placements), then* $\vec{\boldsymbol{p}}$ *is stable if and only if it is rigid.*

4 Forces & Stresses

We represent a set of **applied forces** acting on the structure as a vector of nodal forces:

$$\vec{\boldsymbol{f}} \in \left(\mathbb{R}^3\right)^{\mathcal{N}}. \quad (17)$$

Since the base points are not collinear, any applied force is balanced by reaction forces on them.

Each edge may carry a force; we measure their magnitudes in a convenient way by the **stress vector**

$$\boldsymbol{\omega} \in \mathbb{R}^{\mathcal{E}}. \tag{18}$$

ω_e is the force carried by the edge e divided by the length of e. It is taken to be positive if the force is a tension, so that it generates a force on the node e_α as

$$-\omega_e \left(\vec{\boldsymbol{p}}_{e_\alpha} - \vec{\boldsymbol{p}}_{e_\omega} \right). \tag{19}$$

The totality of the edge-forces acting on a node is then the corresponding entry in the sum

$$-\sum_e \omega_e \boldsymbol{\pi}_e \tag{20}$$

and **(detailed) balance of external and edge-forces** is expressed as

$$\vec{\boldsymbol{f}} = \sum_e \omega_e \boldsymbol{\pi}_e. \tag{21}$$

Recognizing that cables can carry only tension, we call a stress a **proper stress** if

$$\boldsymbol{\omega}_c \geq 0 \quad \text{for all cables } c \tag{22}$$

and call it **strict** if each inequality is strict.

An external force $\vec{\boldsymbol{f}}$ balanced by a stress vector is called **resolvable**. A stress vector $\boldsymbol{\omega}$ is a **prestress** if it resolves $\mathbf{0}$. It is important to note that by (21) a prestress represents a linear dependency amongst the set $\{\boldsymbol{\pi}_e\}$ of edge vectors.

By the definition of the geometric matrix, (21) is

$$\vec{\boldsymbol{f}} = \Pi\, \boldsymbol{\omega}. \tag{23}$$

Thus we can say that the range of Π is the set of all resolvable forces, and its nullspace is the set of all prestresses.

A classical definition of mechanical rigidity, formulated by Maxwell [11] for all-bar trusses, is

Definition 4.1 *A placement of $\mathcal{S}$ is* **statically rigid** *if each applied force is properly resolvable.*

The following was proved for bar-trusses by Maxwell, and generalized to tensegrity structures in [10]

Theorem 4.2 (Roth & Whiteley) *Rigidity is equivalent to statical rigidity.*

The following result from [12], *cf.* [13], is of central importance:

Theorem 4.3 (Whiteley) *Given a placement of $\mathcal{S}$ and $c_o \in \mathcal{C}$, there is an admissible velocity with non-zero stretching in the edge c_o if and only if every proper prestress has $\omega_{c_o} = 0$.*

5 Stability & Rank-deficiency

The T-3 structure is an example of what we choose to call a **Snelson structure**, namely an anchored structure for which

$$\#\mathcal{E} \leq 3\,\#\mathcal{N}. \tag{24}$$

Stable structures with this property are minimal in the sense that they do not have unnecessary edges; Snelson's towers have this property.

Let us consider the possibilities for such a structure. If Π, whose size is $\#\mathcal{E} \times 3\,\#\mathcal{N}$, is of full rank then its null-space is trivial, *i.e.,* it admits only zero prestress. In this case, by Theorem 4.3, it admits a shortening velocity, which, by Theorem 3.3 continues to an admissible motion, and it is not stable. Hence, in a stable placement the geometric matrix must be rank-deficient. By the theorem on rank and nullity and (24) it also must admit a non-trivial left-null-space, that is, the placement is flexible.

Proposition 5.1 *A stable placement for a Snelson structure has a rank-deficient geometric matrix and admits flexure.*

It is possible to strengthen this result for T-3: one can show [9] that stable placements yield geometric matrices of rank-deficiency exactly one.

Thus, to describe stable placements we are led to consider the **rank-deficiency manifolds**:

$$\mathfrak{P}_r = \{\vec{\boldsymbol{p}} \,|\, \mathrm{rank}(\Pi(\vec{\boldsymbol{p}})) = \#\mathcal{E} - r\}, \quad (25)$$
$$1 \le r \le \#\mathcal{E}, \quad (26)$$
$$\mathfrak{P} = \bigcup \mathfrak{P}_r. \quad (27)$$

Since $\mathfrak{P}$ is the set of zeros of the smooth function $\det \Pi(\vec{\boldsymbol{p}})$, it has dimension at most $3\#\mathcal{N} - 1$. Similarly, $\mathfrak{P}_1$ is an open submanifold of $\mathfrak{P}$, *etc.*

We wish to characterize the tangent spaces of the manifolds $\mathfrak{P}_r$. Each is the inverse image under the linear mapping

$$\Pi \,:\, \vec{\boldsymbol{p}} \mapsto [\boldsymbol{\pi}_1(\vec{\boldsymbol{p}}) \ldots \boldsymbol{\pi}_k(\vec{\boldsymbol{p}})] \in \mathbb{R}^{3\mathcal{N}\times\mathcal{E}}, \quad (28)$$

of a particular manifold of matrices. Let us introduce a generic notation:

$$\mathcal{M}_s = \{\, D \in \mathbb{R}^{n\times k} \,|\, \mathrm{rank}(D) = s\}. \quad (29)$$

We continue to write D in terms of its column vectors, staying with our notation

$$D = [\boldsymbol{\pi}_1 \cdots \boldsymbol{\pi}_k]\,. \quad (30)$$

The set $\mathcal{M}_0$ is just an open set in the set of all $n \times k$ matrices (the Stiefel manifold), but each of the smaller ones is a differentiable manifold of reduced dimension (generalized Stiefel manifolds). These were introduced by Milnor [14, 15], but since they do not seem to be well known, we will derive the formulae which we need. The simplest case, when the rank is $k-1$, is a model for the other calculations:

Lemma 5.2 *$\mathcal{M}_{k-1}$ is a differentiable manifold of dimension $(k-1)(n+1)$. Its tangent space at D consists of all $n \times k$ matrices orthogonal to*

$$\vec{\boldsymbol{v}} \otimes \boldsymbol{\omega} \quad (31)$$

where $\boldsymbol{\omega}$ is a non-zero vector in the null space of D and $\vec{\boldsymbol{v}}$ ranges over all vectors in the null space of $D^\top$.

Proof. A matrix $D = [\boldsymbol{\pi}_1 \ldots \boldsymbol{\pi}_k]$ is in the manifold if its column vectors have span of dimension $k-1$ but

$$\boldsymbol{\pi}_1 \wedge \ldots \wedge \boldsymbol{\pi}_k = 0. \quad (32)$$

Consider a path on the manifold passing through D; taking the derivative of (32) at D delivers

$$\sum_{i=1}^{k} \boldsymbol{\pi}_1 \wedge \ldots \wedge \alpha_{\mathbf{i}} \wedge \ldots \wedge \boldsymbol{\pi}_k = 0, \quad (33)$$

where $\alpha_{\mathbf{i}}$, the derivative of $\boldsymbol{\pi}_{\mathbf{i}}(\cdot)$, appears in the ith place in the list. One of the vectors $\boldsymbol{\pi}_l$ can be expressed as a linear combination of the others. To save notation, let us suppose it is the kth:

$$\boldsymbol{\pi}_k = \sum_{j}^{k-1} \mu_j \boldsymbol{\pi}_j, \quad (34)$$

so that the null-vector $\boldsymbol{\omega}$ has entries $[-\mu_1, \ldots, -\mu_n, 1]$. Then

$$\sum_{i}^{k} \sum_{j}^{k-1} \mu_j \boldsymbol{\pi}_1 \wedge \ldots \wedge \alpha_{\mathbf{i}} \wedge \ldots \wedge \boldsymbol{\pi}_j = 0. \quad (35)$$

Note that each exterior product is zero, due to repeated entries, except when $i = j$ or $i = k$. Thus we have

$$\boldsymbol{\pi}_1 \wedge \ldots \wedge \boldsymbol{\pi}_{k-1} \wedge \alpha_{\mathbf{k}} + \sum_{i}^{k-1} \mu_i \boldsymbol{\pi}_1 \wedge \ldots \wedge \alpha_{\mathbf{i}} \wedge \ldots \wedge \boldsymbol{\pi}_i = 0 \quad (36)$$

or

$$(\boldsymbol{\pi}_1 \wedge \ldots \wedge \boldsymbol{\pi}_{k-1}) \wedge \alpha_{\mathbf{k}} + \sum_{i}^{k-1} -(\boldsymbol{\pi}_1 \wedge \ldots \wedge \boldsymbol{\pi}_{k-1}) \wedge (\mu_i \alpha_{\mathbf{i}}) = 0. (37)$$

But this says that

$$(\boldsymbol{\pi}_1 \wedge \ldots \wedge \boldsymbol{\pi}_{k-1}) \wedge \left(\alpha_{\mathbf{k}} - \sum_{i}^{k-1} \mu_i \alpha_{\mathbf{i}} \right) =$$

$$(\boldsymbol{\pi}_1 \wedge \ldots \wedge \boldsymbol{\pi}_{k-1}) \wedge (A\boldsymbol{\omega}) = 0, \tag{38}$$

where $A = [\alpha_1 \ldots \alpha_k]$ is the derivative along the path. (38) means that $A\boldsymbol{\omega}$ is in the span of the other vectors, *i.e.,*, in the range of D, which can be expressed as saying that

$$\vec{\boldsymbol{v}} \cdot A\boldsymbol{\omega} = A \cdot (\vec{\boldsymbol{v}} \otimes \boldsymbol{\omega}) = 0 \tag{39}$$

for all vectors $\vec{\boldsymbol{v}}$ in the null space of $D^\top$. ∎ The proof extends, with only an increase in combinatorial complexity, to each of the manifolds $\mathcal{M}_s$ by considering the smaller submatrices and considering the wedge products of s-lists of column vectors. We obtain

Lemma 5.3 *$\mathcal{M}_s$ is a differentiable manifold of dimension $s(n+k-s)$. Its tangent space at D consists of all A orthogonal to*

$$\boldsymbol{\omega} \otimes \vec{\boldsymbol{v}} \tag{40}$$

where $\boldsymbol{\omega}$ ranges over all vectors in the the null space of D and $\vec{\boldsymbol{v}}$ ranges over all vectors in the null space of $D^\top$.

Now it is a simple matter to specialize this to our manifolds by a retraction from the space of matrices.

Theorem 5.4 *The set $\mathfrak{P}_r$ is a differentiable manifold in $(\mathbb{R}^3)^{\mathcal{N}}$, with tangent space normal to the span of*

$$\sum_e \omega_e B_e \vec{\boldsymbol{v}}, \tag{41}$$

where $\boldsymbol{\omega}$ ranges over all prestresses and $\vec{\boldsymbol{v}}$ ranges over all flexes.

6 Computations

Computation of the stable placements for a given tensegrity structure is an awkward process, involving finding the locations on the appropriate manifold. One means is to use parameterizations or LaGrangian coordinates (*cf. eg.*,[16] and [17]). We take a more direct route in doing calculations for T-3, using the above characterization of the tangent spaces of the manifold $\mathfrak{P}_1$, where its stable placements lie.

By starting at an easily established placement, such as the symmetric one, we march along the manifold by solving numerically a differential equation based on the above calculations. We choose edges in pairs, for example e_1 and e_2, to adjust in length. If we wish to shorten e_2 by an increment δ, we calculate

$$\vec{\boldsymbol{n}} = \sum_e \omega_e B_e \vec{\boldsymbol{v}}, \tag{42}$$

at the current placement (there is only one prestress and one flexure), and then solve the linear equation

$$[\boldsymbol{\pi}_2 \cdots \boldsymbol{\pi}_9 \;\; \vec{\boldsymbol{n}}]^\top \; \vec{\boldsymbol{p}}' = \begin{bmatrix} -\delta \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \tag{43}$$

for the rate of change of placement. Using an RK-4 scheme to execute the step, one arrives at a point nearly on the manifold. Working in *Maple*, it is easy at the end of this step to search locally to find a corrected placement where $\det \Pi$ is (machine-) precisely zero before beginning the next iteration in the process. A few examples of the resulting output are shown in Figures 2–4.

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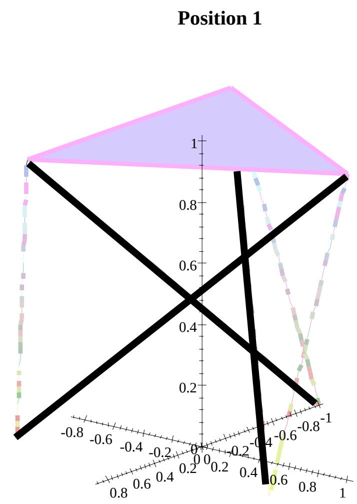

Figure 2: T-3 Structure, Symmetric Position

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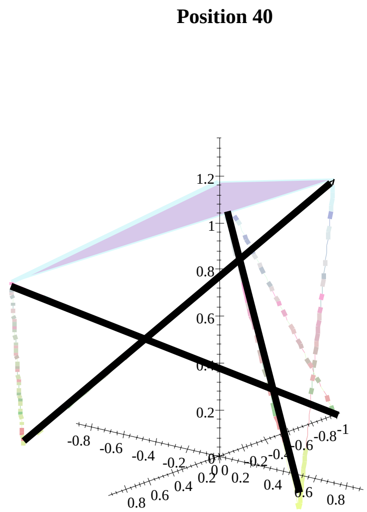

Figure 3: T-3 Structure, Step 40

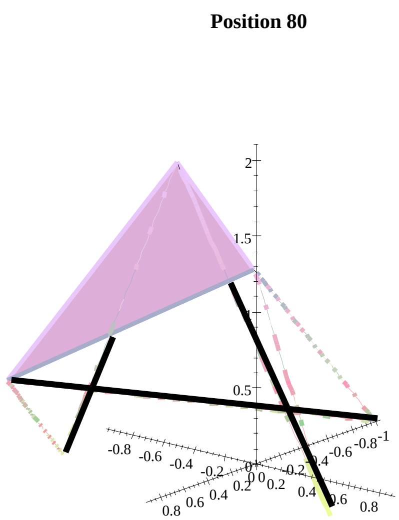

Figure 4: T-3 Structure, Step 80