A Primer on the Mechanics of Tensegrity Structures

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Abstract

We outline the formulation of the theory and the more fundamental results regarding the stability analysis of the class of tensegrity structures: structures which are composed of pin-connected inextensible cables and rigid bars.

Contents

1 Introduction

Following sculptures created by Snelson in 1948, in 1961 Buckminster Fuller patented a class of cable-bar structures which he called tensegrity structures [20, 52]. These consisted of arrangements with bars in compression, no two connected directly, with structural integrity maintained by the tension in the cables. Hence "tension-integrity", compressed to "tensegrity". These structures, remarkable to Fuller for enclosing large volumes of space with minimal weight, are not as well known as his corresponding shell constructions, but offer interest both mathematically and for engineers. (See [16] for a general discussion.) The most famous of these early constructs is Snelson's *Needle Tower* (Fig. 1); the simplest three-dimensional example is the T-3 structure also is shown here (the nomenclature T-3 follows a classification of Kenner [25] of elementary constructs).

The study of these structures remained in the hands of artists and architects until the seventies, when analyses of a generalized form of tensegrity structures appeared in both the mathematical and the engineering literature. The name "tensegrity structure" was extended to include any class of pin-connected frameworks in which some of the frame members are cables, or, complementarily, compression-only struts.

Tensegrity structures offer interesting problems both in structural mechanics, where they have generated one family of literature, *eg*, [5, 48, 7, 27] and in mathematics, with another family, *eg*, [9, 50, 10, 64, 13]. The two

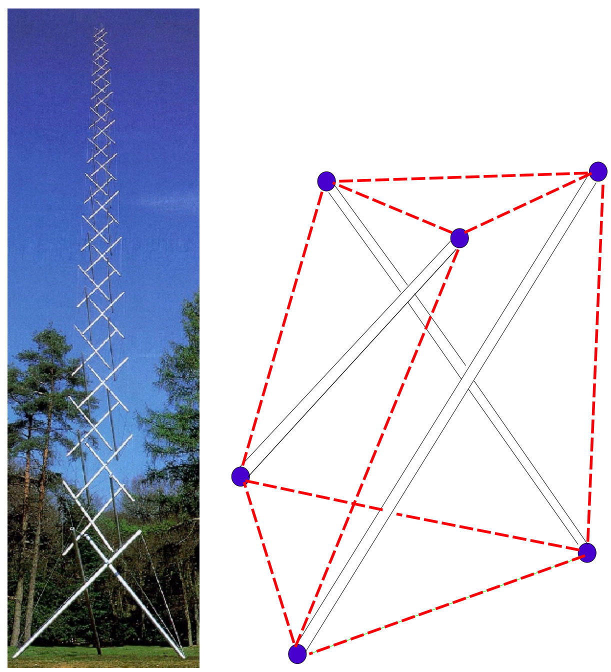

Figure 1: Snelson's Needle Tower and the T-3 Structure

groups of authors unfortunately share only a minimal amount of notation and nomenclature.

This primer collects the results about tensegrity structures which I believe to be most basic and useful. Mostly it is a compendium, but part of it is original, and part semi-original, in that I present new proofs of some established results. It originated as a notebook for me to keep track of results as I learned them; in various error-filled versions, it has circulated amongst my co-workers and friends for the last few years, and finally, I feel compelled to produce a more public and, I hope, more error-free version of the document.

While my personal interest in these structures is their mechanics, I also cannot avoid formulating the subject as a mathematician, and hence the format is rather formal and results are stated and proved as theorems. Also, in light of the aforementioned incompatibility in notations, I presume to introduce my own notations and nomenclature, in particular replacing those nomenclatures from both literatures which I find un-suggestive, misleading, or stilted. Nonetheless, those familiar with either literature will not, I think, find it difficult to pick up the manuscript in the middle and recognize the statements.

A far-from-complete bibliography is appended. In addition to articles referenced directly, I have chosen a collection of references which focus on the theory of tensegrity structures.

2 Kinematics

2.1 Nodes and Edges

Physically, a **structure** is a pin-connected truss. Since the connections are frictionless pins, the elements of the truss pivot freely upon them, and so the elements carry only forces parallel their length. The elements of the truss may be either **bars**, which carry either tension or compression and which can neither extend nor contract, or **cables** which carry only tension and which cannot extend, but may contract (become slack)[1]. The mathematical literature often introduces another sort of element, a *strut* which may endure compression but not tension and cannot contract, but can extend (fall out of the structure) [10, 13]. The mathematical analysis

[1] But in the engineering literature frequently authors consider all or some of the elements to be elastic. Mathematicians also use elastification as a relaxation method to arrive at stable positions of the structure.

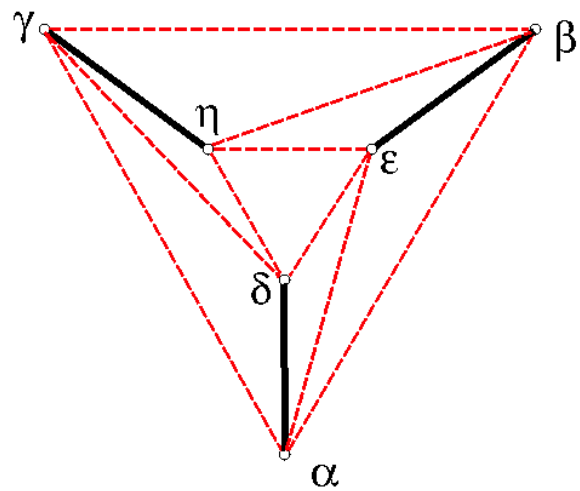

Figure 2: T-3 Structure

is only notationally affected by the presence of struts, which can argue either for their inclusion or their exclusion. Here we choose to consider only cables and bars.

Mathematically, we describe a tensegrity structure as a *connected* bi-graph together with a specification of a set of edge-lengths. The graph consists in a set of **edges** $\mathscr{E}$ and a set of **nodes** $\mathscr{N}$. The edges are divided into two sets, **bars** and **cables**:

$$\mathscr{E} = \mathscr{B} \cup \mathscr{C}. \tag{2.1}$$

Generically, we denote edges by latin minuscules, nodes by greek. The graph of T-3 is shown in Figure 2.

If there are no cables, we may describe the structure as a **bar-truss** and it sometimes is useful to compare the behavior of a structure with cables to the rigidified structure formed by replacing each cable by a bar.

Finally, if $e \in \mathscr{E}$ is an edge we may find it convenient to denote its end-nodes generically as

$$e_\alpha, e_\omega \in \mathscr{N}; \tag{2.2}$$

conversely, if α and β are given nodes, we may choose to call the edge connecting them $\alpha\beta$. The use of these two conventions interchangeably proves convenient.

2.2 Vector Spaces

We shall have occasion to deal with several vector spaces based on the sets above. In particular, we use the space

$$\mathbb{R}^{3\mathcal{N}} := \left(\mathbb{R}^3\right)^{\mathcal{N}}, \tag{2.3}$$

whose elements will be denoted by blackboard-bold symbols like $\mathbb{p}$ with components $\mathbb{p}_\alpha \in \mathbb{R}^3$, and the space

$$\mathbb{R}^{\mathcal{E}}, \tag{2.4}$$

whose elements we denote by bold greek letters, as $\boldsymbol{\omega}$.

We introduce two convenient conventions. First, given a linear map $L \in \mathrm{gl}(\mathbb{R}^3)$, we use the same symbol to denote the product map on $\mathbb{R}^{3\mathcal{N}}$ given as

$$L : \mathbb{R}^{3\mathcal{N}} \mapsto \mathbb{R}^{3\mathcal{N}}; \quad (L\mathbb{v})_\alpha = L(\mathbb{v}_\alpha). \tag{2.5}$$

Of course the re-interpreted map is linear. Similarly, given a vector $\mathbf{v} \in \mathbb{R}^3$ we use the same symbol for its $\#\mathcal{N}$-fold product in $\mathbb{R}^{3\mathcal{N}}$. For example, if $\mathbb{w} \in \mathbb{R}^{3\mathcal{N}}$, we identify $\mathbb{w} + \mathbf{v}$, and, in particular, $\mathbb{w} + \mathbb{w}_\delta$ by

$$(\mathbb{w} + \mathbf{v})_\alpha = \mathbb{w}_\alpha + \mathbf{v} \quad \text{or} \quad (\mathbb{w} + \mathbb{w}_\delta)_\alpha = \mathbb{w}_\alpha + \mathbb{w}_\delta\,. \tag{2.6}$$

2.3 Placements and Motions

A **placement** of the graph $\mathcal{S}$ is an mapping of each node into $\mathbb{R}^3$. It is most convenient to describe the placement as a map into $\mathbb{R}^{3\mathcal{N}}$:

$$\mathbb{p} \in \mathbb{R}^{3\mathcal{N}}. \tag{2.7}$$

We choose to ignore certain impractical special cases as follows:

Remark 2.1. *We always will assume not only that $\mathbb{p}$ is injective, but also that the placement is not (totally) collinear .*

Physically it is clear that the lengths of the edges are fundamental. We introduce lengths in a mathematically convenient way through the (one-half-squared-) **length-map**

$$\boldsymbol{\lambda} : \mathbb{R}^{3\mathcal{N}} \to \mathbb{R}^{\mathcal{E}} \tag{2.8a}$$

$$(\boldsymbol{\lambda}(\mathbb{p}))_e = \frac{1}{2}||\mathbb{p}_{e_\alpha} - \mathbb{p}_{e_\omega}||^2. \tag{2.8b}$$

We often suppose that a set of **edge lengths**

$$\mathbf{\Lambda} \in \mathbb{R}^{\mathscr{E}} \tag{2.9}$$

is assigned, normally as calculated from a base placement. Then, given $\mathbf{\Lambda}$, a placement $\mathbb{p}$ is said to be admissible if

$$\boldsymbol{\lambda}_b(\mathbb{p}) = \mathbf{\Lambda}_b, \ \ \forall\, b \in \mathscr{B} \text{ and} \tag{2.10a}$$

$$\boldsymbol{\lambda}_c(\mathbb{p}) \leq \mathbf{\Lambda}_c, \ \ \forall\, c \in \mathscr{C}. \tag{2.10b}$$

Given a base placement $\mathbb{p}$ and corresponding set of edge lengths $\lambda(\mathbb{p})$ we let $\mathrm{Adm}(\mathbb{p})$ denote the set of **admissible placements** relative $\mathbb{p}$. One special subset of $\mathrm{Adm}(\mathbb{p})$ are the placements obtained by a rigid motion of the given one. Such a rigid movement is described by a proper orthogonal linear map on $\mathbb{R}^3$ plus a translation. (We restrict to proper rotations since improper ones often cannot be effected for a structure without passing one edge through another.) This set, the **rigidly equivalent placements**, are

$$\mathrm{Euc}(\mathbb{p}) = \{\, Q\,\mathbb{p} + \mathbf{r} \,|\, Q \in \mathrm{SO}(3),\ \ \mathbf{r} \in \mathbb{R}^3 \,\} \subset \mathrm{Adm}(\mathbb{p})\,. \tag{2.11}$$

Since the operations $\mathbb{p} \mapsto Q\mathbb{p} + \mathbf{r}$ represent a group action on $\mathbb{R}^{3\mathcal{N}}$, the sets defined as in (2.11) form a partition of $\mathbb{R}^{3\mathcal{N}}$ into equivalence classes. Physically, it is clear that in our analysis of the properties of a structure we should identify placements equivalent in this sense. We will consider this systematically in Section 3.1.

It is worth noting that Q when regarded as a linear map on $\mathbb{R}^{3\mathcal{N}}$ still is orthogonal.

A **motion** away from a given placement $\mathbb{p}$ is an analytic one-parameter family of placements

$$t \mapsto \mathbb{q}(t); \qquad \mathbb{q}(0) = \mathbb{p}, \tag{2.12}$$

defined on a half-neighborhood of 0. It is said to be an **admissible motion** if $\mathbb{q}(t) \in \mathrm{Adm}(\mathbb{p})$ for all t and a **rigid motion** if $\mathbb{q}(t) \in \mathrm{Euc}(\mathbb{p})$ for all t.

A motion generates an initial velocity vector

$$\mathbb{v} = \dot{\mathbb{q}}(0) \in \mathbb{R}^{3\mathcal{N}}. \tag{2.13}$$

Generalizing, we may call any element in $\mathbb{R}^{3\mathcal{N}}$ a **velocity**. In the literature a velocity also is called an **infinitesimal displacement**.

If $\mathbb{q}(\cdot)$ is a motion initiating at $\mathbb{p}$, then the rate of change of the length function for edge e at time 0 is given by

$$\frac{d}{dt}\,\lambda_e(\mathbb{q})(0) = \nabla\boldsymbol{\lambda}_e(\mathbb{p}) \cdot \dot{\mathbb{q}}(0) = (\,\mathbb{p}_{e_\alpha} - \mathbb{p}_{e_\omega}\,) \cdot (\dot{\mathbb{q}}_{e_\alpha} - \dot{\mathbb{q}}_{e_\omega})\,. \tag{2.14}$$

Generalizing, we assign to any placement and any velocity the **stretching vector**

$$\boldsymbol{\epsilon}(\mathbb{p}, \mathbb{v}) \in \mathbb{R}^{\mathscr{E}}, \tag{2.15}$$

calculated as

$$\begin{aligned} \boldsymbol{\epsilon}_e &= \nabla \boldsymbol{\lambda}_e(\mathbb{p}) \cdot \mathbb{v} \\ &= \left(\mathbb{p}_{e_\alpha} - \mathbb{p}_{e_\omega}\right) \cdot \left(\mathbb{v}_{e_\alpha} - \mathbb{v}_{e_\omega}\right) . \end{aligned} \tag{2.16}$$

$\boldsymbol{\epsilon}_e$ gives the rate of lengthening of the edge e times the length of the edge.

At this stage, it is convenient to introduce the **strain cone** for the structure as the convex closed cone

$$\mathfrak{E} = \{\boldsymbol{\nu} \in \mathbb{R}^{\mathscr{E}} \mid \boldsymbol{\nu}_b = 0 \;\; \forall b \in \mathscr{B}, \text{ and } \boldsymbol{\nu}_c \leq 0 \;\; \forall c \in \mathscr{C}\}. \tag{2.17}$$

Its polar (convex closed) cone is the **stress cone**

$$\begin{aligned} \mathfrak{S} = \mathfrak{E}^o &= \{\boldsymbol{\nu} \in \mathbb{R}^{\mathscr{E}} \mid \boldsymbol{\nu} \cdot \boldsymbol{\mu} \leq 0 \;\; \forall \boldsymbol{\mu} \in \mathfrak{E}\} \\ &= \{\boldsymbol{\mu} \in \mathbb{R}^{\mathscr{E}} \mid \boldsymbol{\mu}_c \geq 0 \;\; \forall c \in \mathscr{C}\} \end{aligned} \tag{2.18}$$

This allows us, for example, to express the condition of admissibility (2.10) more compactly as

$$\boldsymbol{\lambda} \in \boldsymbol{\Lambda} + \mathfrak{E}, \tag{2.19}$$

but it is particularly use in discussing velocities (and, later, stresses). Thus, $\boldsymbol{\epsilon}$ is compatible with (2.10) if and only if $\boldsymbol{\epsilon} \in \mathfrak{E}$, so we define the set of **admissible velocities** for $\mathbb{p}$ to be

$$\mathbb{V}(\mathbb{p}) = \{\mathbb{v} \in \mathbb{R}^{3\mathscr{N}} \mid \boldsymbol{\epsilon}(\mathbb{p}, \mathbb{v}) \in \mathfrak{E}\}, \tag{2.20}$$

and distinguish a subset of $\mathbb{V}$, the **null velocities**

$$\mathbb{V}_0(\mathbb{p}) = \{\mathbb{v} \in \mathbb{R}^{3\mathscr{N}} \mid \boldsymbol{\epsilon}(\mathbb{p}, \mathbb{v}) = \mathbf{0}\}, \tag{2.21}$$

as those velocities which do not change any lengths. The set $\mathbb{V}$ is a closed convex cone which includes the subspace $\mathbb{V}_0$.

Finally, the set of all initial velocities which can be generated by a rigid-body motion away from $\mathbb{p}$ is the six-dimensional subspace of **rigid velocities**,

$$\begin{aligned} \mathscr{R}(\mathbb{p}) := \big\{\mathbb{v} \mid \mathbb{v} = W\mathbb{p} + \mathbf{v} \;\; &\text{for some skew linear map } W \text{ of } \mathbb{R}^3, \\ &\text{and vector } \mathbf{v} \in \mathbb{R}^3\big\} . \end{aligned} \tag{2.22}$$

Of course this is a subspace of $\mathbb{V}_0$.

The vector $\nabla\boldsymbol{\lambda}_e$ which appears in (2.16) will be called an **edge vector**; it has the form

$$\boldsymbol{\pi}_e(\mathbb{p}) = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \mathbb{p}_{e_\alpha} - \mathbb{p}_{e_\omega} \\ 0 \\ \cdot \\ 0 \\ \mathbb{p}_{e_\omega} - \mathbb{p}_{e_\alpha} \\ 0 \\ \vdots \\ 0 \end{bmatrix} \in \mathbb{R}^{3\mathcal{N}}, \tag{2.23}$$

Here, as suggested by the notation, the only non-zero entries occur in the e_α and e_ω (end-node) slots, respectively.

Each edge vector is a linear function of $\mathbb{p}$ and we may write

$$\boldsymbol{\pi}_e = B_e\,\mathbb{p} \tag{2.24}$$

for each e in $\mathscr{E}$. The linear transformations, or **edge maps** B_e, map $\mathbb{R}^{3\mathcal{N}}$ into $\mathbb{R}^{3\mathcal{N}}$, are symmetric, positive, and obey

$$(B_e)^2 = 2\,B_e. \tag{2.25}$$

An important property is that they commute with linear maps of $\mathbb{R}^3$, *ie,*

$$B_e\,L = L\,B_e. \tag{2.26}$$

Similarly, given a vector $\mathbf{v} \in \mathbb{R}^3$, it is easy to see that

$$B_e\,\mathbf{v} = \mathbf{0} \tag{2.27}$$

(recall the convention that $\mathbf{v}$ also denotes the repeated version in $\mathbb{R}^{3\mathcal{N}}$). Alternative forms for the length of edge e in the placement $\mathbb{p}$ then are

$$\begin{aligned} \lambda_e(\mathbb{p}) &= \frac{1}{4}\,\boldsymbol{\pi}_e \cdot \boldsymbol{\pi}_e, \\ &= \frac{1}{4}\,B_e\mathbb{p} \cdot B_e\mathbb{p} = \frac{1}{2}\,\mathbb{p} \cdot B_e\mathbb{p} \end{aligned} \tag{2.28}$$

and we rediscover the fundamental relation (2.23) as

$$\nabla \boldsymbol{\lambda}_e = B_e \mathbb{p} = \boldsymbol{\pi}_e\,, \tag{2.29}$$

and see that $B_e = \nabla^2 \boldsymbol{\lambda}_e$.

Next we use the edge vectors as column vectors to construct the **geometric matrix**

$$\Pi = \left[\cdots \boldsymbol{\pi}_e \cdots\right] : \mathbb{R}^{\mathscr{E}} \to \mathbb{R}^{3\mathscr{N}}. \tag{2.30}$$

This matrix, or its transpose, usually is called the **rigidity matrix** in the mathematical literature. It is related by a factor of an edge-lengths in each column to the **structural matrix** preferred in the engineering literature.

Then Π is useful for calculations such as

$$\boldsymbol{\epsilon} = \Pi^\top \mathbb{v}. \tag{2.31}$$

In particular,

$$\mathbb{V}_0 = \mathrm{Null}(\Pi^\top); \quad \Pi^\top \mathbb{V} \subset \mathfrak{E}. \tag{2.32}$$

Similarly, we can now concatenate the relations (2.29) to obtain

$$\Pi = \nabla \boldsymbol{\lambda}^\top\,. \tag{2.33}$$

Finally, we collect some computations for rigidly equivalent placements.

Proposition 2.2. *Given a placement* $\mathbb{p}$*, an orthogonal* Q *and a vector* $\mathbf{r}$ *in* $\mathbb{R}^3$

(a) $\boldsymbol{\lambda}(Q\mathbb{p} + \mathbf{r}) = \boldsymbol{\lambda}(\mathbb{p})$

(b) $\boldsymbol{\pi}_e(Q\mathbb{p} + \mathbf{r}) = Q\,\boldsymbol{\pi}_e(\mathbb{p})$ *for all* $e \in \mathscr{E}$

(c) $\Pi(Q\mathbb{p} + \mathbf{r}) = Q\,\Pi(\mathbb{p})$

(b) $\epsilon(Q\mathbb{p} + \mathbf{r}, Q\,\mathbb{v}) = \epsilon(\mathbb{p}, \mathbb{v})$ *for all* $\mathbb{v} \in \mathbb{R}^{3\mathscr{N}}$

(b) $\mathbb{V}(Q\mathbb{p} + \mathbf{r}) = Q\,\mathbb{V}(\mathbb{p})$

(b) $\mathbb{V}_0(Q\mathbb{p} + \mathbf{r}) = Q\,\mathbb{V}_0(\mathbb{p})$

(b) $\mathscr{R}(Q\mathbb{p} + \mathbf{r}) = Q\,\mathscr{R}(\mathbb{p})$

The proof all are immediate, once we recognize that for any linear map L of $\mathbb{R}^3$ and vector $\mathbf{v}$ in $\mathbb{R}^3$

$$B_e(L\,\mathbb{w} + \mathbf{r}) = L\,B_e(\mathbb{w})$$

for all vectors $\mathbb{w}$.

2.4 Reduced Kinematics

The previous constructions lead to large sparce matrices which are notationally cumbersome and may be computationally awkward. A more compact notation can be effected by the identification

$$\mathbb{R}^{3\mathcal{N}} \leftrightarrow \mathbb{R}^{\mathcal{N}} \otimes \mathbb{R}^{3} \tag{2.34}$$

created by the bilinear map

$$\mathbb{R}^{\mathcal{N}} \times \mathbb{R}^{3} \ni ([r_\alpha]_{\alpha\in\mathcal{N}}, \mathbf{u}) \mapsto [r_\alpha \mathbf{u}]_{\alpha\in\mathcal{N}} \in \mathbb{R}^{\mathcal{N}} \,. \tag{2.35}$$

This implies the identification of the tensor product as

$$\mathbb{r} \leftrightarrow \sum_{\alpha\in\mathcal{N}} \rho_\alpha \otimes \mathbb{r}_\alpha \,, \tag{2.36}$$

where (ρ_α) is the standard orthonormal basis of $\mathbb{R}^{\mathcal{N}}$.

Then it is easy to see that

$$B_e \leftrightarrow C_e \otimes \mathbb{1} \tag{2.37}$$

where $\mathbb{1}$ is the identity on $\mathbb{R}^3$ and C_e is the **edge-connection operator**, an automorphism of $\mathbb{R}^{\mathcal{N}}$ with standard matrix

$$\begin{array}{cc} & \begin{array}{ccccccc} & & e_\alpha & & e_\omega & & \end{array} \\ \begin{array}{c} \\ e_\alpha \\ \\ e_\omega \\ \\ \end{array} & \begin{bmatrix} 0 & \cdot & \cdot & \cdot & \cdot & \cdot & 0 \\ 0 & \cdot & 1 & \cdot & -1 & \cdot & 0 \\ 0 & \cdot & \cdot & \cdot & \cdot & \cdot & 0 \\ 0 & \cdot & -1 & \cdot & 1 & \cdot & 0 \\ 0 & \cdot & \cdot & \cdot & \cdot & \cdot & 0 \end{bmatrix} \end{array} \,. \tag{2.38}$$

The operators C_e are symmetric, positive, and $C_e^2 = 2C_e$.

Using these we can re-express the edge vector, for example, as

$$\boldsymbol{\pi}_e(\mathbb{p}) \leftrightarrow \sum_{\alpha\in\mathcal{N}} (C_e\, \rho_\alpha) \otimes p_\alpha \,, \tag{2.39}$$

and formally, letting $\{\gamma_e \mid e \in \mathscr{C}\}$ be the standard unit vectors in $\mathbb{R}^{\mathscr{C}}$, we can rewrite the geometric matrix as

$$\Pi \leftrightarrow \sum_{e\in\mathscr{E}} \sum_{\alpha\in\mathcal{N}} (C_e\, \rho_\alpha) \otimes \mathbb{p}_\alpha \otimes \gamma_e \,. \tag{2.40}$$

Finally, the tensor identification enables a more formal version of our identification of an automorphism L of $\mathbb{R}^3$ with one of $\mathbb{R}^{3\mathcal{N}}$ as

$$\mathbb{1} \otimes L \,. \tag{2.41}$$

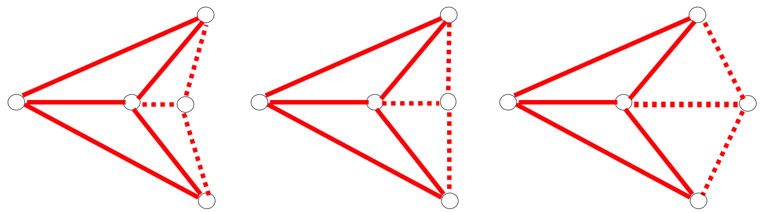

Figure 3: Two Dimensional Examples, Resp. Rigid, Stable and Unstable

3 Rigidity and Stability

Finally, we can define the fundamental notions of the theory:

Definition 3.1. *A placement $\mathbb{p}$ is* **rigid** *if the only velocities admissible for $\mathbb{p}$ are rigid velocities.*

Definition 3.2. *A placement $\mathbb{p}$ is* **stable** *if the only admissible motions from $\mathbb{p}$ are rigid-body motions.*

Figure(3) illustrates, in $\mathbb{R}^2$, the difference of the two concepts: the first of the frames is rigid, the second is stable but not rigid, since there is an admissible velocity (which does not continue to an admissible motion), while the third clearly is unstable.

A useful restatement of stability is the following.

Proposition 3.3. *$\mathbb{p}$ is stable if and only if there is a neighborhood of $\mathbb{p}$ in which the only admissible placements are rigid-equivalents of $\mathbb{p}$.*

Proof. Clearly if there can be no non-equivalent placements in a neighborhood there can be no family of such placements.

Conversely, let us suppose that every neighborhood of the placement $\mathbb{p}$ includes a non-equivalent admissible placement and show that we can produce a one-parameter family of placements originating at $\mathbb{p}$. Following Roth and Whiteley [50], who adapt an argument for bar-structures to this case, we note that the set of all non-rigid admissible placements can be described as an algebraic set by defining

$$\mathscr{D} = \Big\{ (\mathbb{q}, \kappa) \in \mathbb{R}^{3\mathcal{N}} \times \mathbb{R}^{\mathscr{C}} \mid \forall b \in \mathscr{B},\ \ c \in \mathscr{C} \\ \lambda_b(\mathbb{q}) = \lambda_b(\mathbb{p}),\ \lambda_c(\mathbb{q}) + \kappa^2 = \lambda_c(\mathbb{p}) \Big\} \setminus \mathrm{Euc}(\mathbb{p}), \quad (3.1)$$

since the length functions are quadratic and elements of Euc are linear in $\mathbb{q}$. Then $\mathbb{q}$ is admissible exactly when there is a number κ such that $(\mathbb{q}, \kappa) \in \mathscr{D}$.

Then the continuity of the length function ensures that each neighborhood of $(\mathbb{p}, 0)$ includes an element of $\mathscr{D}$, and the curve-selection lemma of Milnor [37] guarantees that there is an analytic path in $\mathscr{D}$ originating at $(\mathbb{p}, 0)$. This is the required motion. □

Remarks:

- Nomenclature varies in the literature; in particular, structures which we call stable would be called *rigid* in the mathematical literature, and those which we call rigid are said to be *first-order rigid*.
- Glück in [21] observes that the degree of assumed smoothness of motions is irrelevant in determining stability; hence we have chosen analyticity as a condition.

3.1 Fixing the structure

Intuitively, it is clear that all considerations of rigidity and stability should be independent of superposed rigid motions. Since many constructions are greatly complicated by the possible presence of these added motions, it is useful to eliminate them explicitly by choosing a particular class of representative elements for the equivalence classes Euc($\mathbb{p}$).

Our method of choice is outlined as follows:

Proposition 3.4. *Let the structure have at least one bar $\alpha\beta$, and have at least two edges. Given a placement $\mathbb{p}$ of the structure there is exactly one $\mathbb{p}^* \in Euc(\mathbb{p})$ which has α at the origin, β lying on the positive x-axis and has a specified node, γ, not collinear with β in the original placement, constrained to lie in the $y > 0$ half of the x-y plane.*

Proof. We introduce the right-handed ordered triple of orthonormal vectors $\mathbf{e}$, $\mathbf{f}$, $\mathbf{g}$ such that

$$\mathbb{p}_\beta = \mathbb{p}_\alpha + \mu\mathbf{e}, \quad \mathbb{p}_\gamma = \mathbb{p}_\alpha + \nu\mathbf{e} + \phi\mathbf{f}.$$

We then construct the proper orthogonal $Q = \mathbf{e} \otimes \mathbf{e}_x + \mathbf{f} \otimes \mathbf{e}_y + \mathbf{g} \otimes \mathbf{e}_z$, and set $\mathbf{r} = -Q\mathbb{p}_\alpha$. It follows that

$$\mathbb{p}^* = Q\mathbb{p} + \mathbf{r} = Q(\mathbb{p} - \mathbb{p}_\alpha) \tag{3.2}$$

has the required properties. Clearly, it is unique in Euc($\mathbb{p}$), as the rigid motion which carries the first such placement to the next must leave α and β on the positive x–axis; a rotation which carries γ into a new position must leave the axis fixed and hence move γ from the plane. □

Definition 3.5. *For a given choice of nodes* α, β, γ*, the class of all placements having the properties in Proposition 3.4 is denoted* **Rep**.

Proposition 3.6. Rep *is an affine subspace of* $\mathbb{R}^{3\mathcal{N}}$ *with tangent space*

$$\mathcal{U} = \{\mathbb{v}^* \in \mathbb{R}^{3\mathcal{N}} \mid \mathbb{v}^*_\alpha = \mathbf{0},\ \mathbb{v}^*_\beta \cdot \mathbf{e}_y = \mathbb{v}^*_\beta \cdot \mathbf{e}_z = 0,\ \mathbb{v}^*_\gamma \cdot \mathbf{e}_z = 0\}. \tag{3.3}$$

The subspace $\mathcal{U}$ is the space of velocities which can be generated by motions which stay within Rep. It is complementary to the spaces of rigid velocities:

Proposition 3.7. *For any* $\mathbb{p}^*$ *in* Rep $\mathbb{R}^{3\mathcal{N}} = \mathcal{U} \oplus \mathcal{R}(\mathbb{p}^*)$*, ie,*

$$\mathcal{U} + \mathcal{R}(\mathbb{p}^*) = \mathbb{R}^{3\mathcal{N}}, \quad \mathcal{U} \cap \mathcal{R}(\mathbb{p}^*) = \{\mathbf{0}\}. \tag{3.4}$$

Proof. Let $\mathbb{v} \in \mathbb{R}^{3\mathcal{N}}$ be given. We seek a decomposition $\mathbb{v} = \mathbb{u} + \mathbb{r}$. To ensure that $\mathbb{u} \in \mathcal{U}$ we must construct the rigid velocity with

$$\begin{aligned} \mathbb{r}_\alpha &= \mathbb{v}_\alpha \\ (\mathbb{r}_\beta)_y \mathbf{e}_y + (\mathbb{r}_\beta)_z \mathbf{e}_z &= (\mathbb{v}_\beta)_y \mathbf{e}_y + (\mathbb{v}_\beta)_z \mathbf{e}_z \\ (\mathbb{r}_\gamma)_z \mathbf{e}_z &= (\mathbb{v}_\gamma)_z \mathbf{e}_z. \end{aligned} \tag{3.5}$$

But $\mathbb{r}$ must have the form $\mathbb{r} = W\mathbb{p}^* + \mathbf{v}$. The skew mapping W can be expressed as

$$W = w_z\, \mathbf{e}_x \wedge \mathbf{e}_y + w_y\, \mathbf{e}_x \wedge \mathbf{e}_z + w_x\, \mathbf{e}_y \wedge \mathbf{e}_z. \tag{3.6}$$

To obey the first of (3.5), since $\mathbb{p}^*_\alpha = \mathbf{0}$ we must take $\mathbf{v} = \mathbb{v}_\alpha$. Next, $\mathbb{p}^*_\beta = \lambda \mathbf{e}_x$ with $\lambda > 0$ so we must have

$$\mathbb{r}_\beta = \lambda W \mathbf{e}_x + \mathbb{v}_\alpha = -\lambda w_z \mathbf{e}_y - \lambda w_y \mathbf{e}_z + \mathbb{v}_\alpha. \tag{3.7}$$

Thus we are left with

$$\begin{aligned} -\lambda w_z + (\mathbb{v}_\alpha)_y &= (\mathbb{v}_\beta)_y \\ -\lambda w_y + (\mathbb{v}_\alpha)_z &= (\mathbb{v}_\beta)_z \end{aligned} \tag{3.8}$$

which yields a unique pair w_z and w_y. Similarly, $\mathbb{p}^*_\gamma = \mu \mathbf{e}_x + \nu \mathbf{e}_y$, with $\nu > 0$, so

$$\mathbb{r}_\gamma = \nu w_z \mathbf{e}_x - \mu w_z \mathbf{e}_y - (\mu w_y + \nu w_x)\mathbf{e}_z + \mathbb{v}_\alpha, \tag{3.9}$$

leading to

$$-\mu w_y - \nu w_x + (\mathbb{v}_\alpha)_z = (\mathbb{v}_\gamma)_z. \tag{3.10}$$

Given the previous computations, this determines the value of w_x. Thus the decomposition exists and is unique. ◻

Corollary 3.8. *All derivatives of a motion in* Rep *lie in the subspace* $\mathcal{U}$.

Next we consider stability and rigidity. We can apply the construction in the proof of Proposition 3.4 to map motions $\mathbb{q}(t)$ to equivalent motions in Rep. For each t we can construct a $Q(t)$ to obtain

$$\mathbb{q}^*(t) = Q(t)(\mathbb{q}(t) - \mathbb{q}_\alpha(t)) \in \mathrm{Rep}. \tag{3.11}$$

By the construction, the function $Q(t)$ is smooth and hence the new motion is smooth.

For later uses, we calculate the initial derivatives of the new motion. Given $\mathbb{v}$ and $\mathbb{a}$ for $\mathbb{p}$, there are skew linear maps W and S such that the initial velocity $\mathbb{v}^*$ and initial acceleration $\mathbb{a}^*$ for $\mathbb{p}^*$ have the forms

$$\mathbb{v}^* = WQ(\mathbb{p} - \mathbb{p}_\alpha) + Q(\mathbb{v} - \mathbb{v}_\alpha) = W\mathbb{p}^* + Q(\mathbb{v} - \mathbb{v}_\alpha), \tag{3.12}$$

and

$$\begin{aligned} \mathbb{a}^* &= (S + W^2)Q(\mathbb{p} - \mathbb{p}_\alpha) + 2WQ(\mathbb{v} - \mathbb{v}_\alpha) + Q(\mathbb{a} - \mathbb{a}_\alpha), \\ &= (S + W^2)\mathbb{p}^* + 2WQ(\mathbb{v} - \mathbb{v}_\alpha) + Q(\mathbb{a} - \mathbb{a}_\alpha), \end{aligned} \tag{3.13}$$

where $Q = Q(0)$. (Recall the convention about insertions into the larger space: $\mathbb{v} - \mathbb{v}_\alpha$ represents the difference of $\mathbb{v}$ with the vector in $\mathbb{R}^{3\mathcal{N}}$ all of whose entries are $\mathbb{v}_\alpha$.) The relations can be inverted to give $\mathbb{v}$ and $\mathbb{a}$ in term of $\mathbb{v}^*$ and $\mathbb{a}^*$.

Now we verify the expected criteria for stability and rigidity in Rep, and relate these to those for equivalent general placements.

Proposition 3.9. *A placement* $\mathbb{p}^*$ *in* Rep *is stable if and only if there are no admissible motions starting from the placement and remaining in* Rep.

Proof. If $\mathbb{p}^*$ is stable, then the only admissible motions starting from $\mathbb{p}^*$ are rigid motions. But no rigid motions stay in Rep.

If $\mathbb{p}^*$ is not stable, there exists a non-rigid admissible $\mathbb{q}(t)$ originating at $\mathbb{p}^*$. But we can use (3.11) to construct an equivalent motion $\mathbb{q}^*(t)$ in Rep originating at $\mathbb{p}^*$ ($Q(0) = I$). It is admissible, since the rigid mappings used in the construction all conserve lengths. Existence of this motion will serve to show $\mathbb{p}^*$ unstable, once we verify that the construction does not create a constant-valued motion. But were it constant,

$$\mathbb{q}^*(t) = Q(t)\left(\mathbb{q}(t) - \mathbb{q}_\alpha(t)^\dagger\right) = \mathbb{p}^* \tag{3.14}$$

and hence

$$\mathbb{q}(t) = Q(t)^\top \mathbb{p}^* + \mathbb{q}_\alpha(t)^\dagger \tag{3.15}$$

would be a rigid motion. □

Proposition 3.10. *A placement $\mathbb{p}^*$ in Rep is rigid if and only if it has no non-trivial admissible velocities in $\mathcal{U}$.*

Proof. If it is rigid then the only admissible velocities are in $\mathcal{R}$, which is complementary to $\mathcal{U}$.

If it is not rigid, it has a non-rigid admissible velocity. But this velocity has a decomposition $\mathbb{v} + \mathbb{r}$ with $\mathbb{v} \in \mathcal{U}$ not zero. Then $\Pi^\top(\mathbb{v} + \mathbb{r}) = \Pi^\top \mathbb{v} \in \mathfrak{E}$ so that $\mathbb{v}$ is a non-trivial admissible velocity in $\mathcal{U}$. □

Finally, we obtain the desired reduction which removes the rigid-body-motion from our tests for stability and rigidity.

Proposition 3.11. *A placement $\mathbb{p}$ is stable if and only if its equivalent $\mathbb{p}^* \in$ Rep is stable.*

Proof. Suppose the motion $\mathbb{q}(t)$ starts at $\mathbb{p}$ and is admissible but not a rigid body motion. We convert it into a motion in Rep from $\mathbb{p}^*$. This motion is admissible; by an the argument used in Proposition 3.9, since $\mathbb{q}$ is not rigid $\mathbb{q}^*$ is non-trivial.

Conversely, given a motion from $\mathbb{p}^*$ we use the inverse of the other construction to find a motion from $\mathbb{p}$. That motion is non-trivial, since $\mathbb{q}^*$ is not rigid. □

Proposition 3.12. *A placement $\mathbb{p}$ is rigid if and only if its equivalent $\mathbb{p}^* \in$ Rep is rigid.*

Proof. Choose the orthogonal Q carrying $\mathbb{p}$ to $\mathbb{p}^*$. By Proposition 2.2

$$\mathbb{V}(\mathbb{p}^*) = Q\mathbb{V}(\mathbb{p}) \quad \text{and} \quad \mathcal{R}(\mathbb{p}^*) = Q\mathcal{R}(\mathbb{p}), \tag{3.16}$$

Thus the set of admissible velocities for either placement consists only of rigid velocities if and only if the same is true for the other. □

The fact that a rigid placement is stable is not entirely obvious in the general case: one has to eliminate the possibility that a rigid velocity might extend to a motion which is not a rigid body motion (*cf* [13]). But in Rep this is trivial: if a motion from the placement exists then it must have a non-zero derivative, and this derivative must lie in $\mathcal{U}$, so it is not in $\mathcal{R}$.

Proposition 3.13. *A placement which is rigid is stable.*

3.2 Further Characterization of Rigidity

An important property of Π is

Proposition 3.14. *For each edge e*

$$\boldsymbol{\pi}_e(\mathbb{p}) \in \mathscr{R}(\mathbb{p})^{\perp}. \tag{3.17}$$

Proof. Given e and any rigid velocity $W\mathbb{p} + \mathbf{v}$, since B_e is symmetric and $B_e^2 = 2B_e$,

$$\begin{aligned} 2\,\boldsymbol{\pi}_e \cdot (W\mathbb{p} + \mathbf{v}) &= 2\,B_e\mathbb{p} \cdot (W\mathbb{p} + \mathbf{v}) = 2\,B_e\mathbb{p} \cdot W\mathbb{p} \\ &= B_e^2\mathbb{p} \cdot W\mathbb{p} = B_e\mathbb{p} \cdot \mathscr{B}_e W\mathbb{p} \\ &= B_e\mathbb{p} \cdot WB_e\mathbb{p} = 0, \end{aligned} \tag{3.18}$$

since W is skew. □

Corollary 3.15. *The range of* $\Pi(\mathbb{p})$ *is in* $\mathscr{R}(\mathbb{p})^{\perp}$.

This, since the domain of the matrix is $\mathbb{R}^{\mathscr{E}}$, leads to an important observation.

Remark 3.16. *The dimension of the range of* Π *is no greater than* $3(\#\mathscr{N}) - 6$ *and the dimension of its domain is* $\#\mathscr{E}$.

Recall that the placement is rigid exactly when

$$\mathbb{V} = \mathscr{R}. \tag{3.19}$$

We characterize this through use of the following lemma.

Lemma 3.17. *The polar of* $\mathbb{V}$ *is* $\mathbb{V}^o = \Pi\,\mathfrak{S}$.

Proof. Consider

$$\begin{aligned} (\Pi\,\mathfrak{S})^o &= \{\, \mathbb{v} \mid \mathbb{v} \cdot \Pi\boldsymbol{\mu} \le 0 \ \ \forall \boldsymbol{\mu} \in \mathfrak{S} \,\} \\ &= \{\, \mathbb{v} \mid \Pi^{\top}\mathbb{v} \cdot \boldsymbol{\mu} \le 0 \ \ \forall \boldsymbol{\mu} \in \mathfrak{S} \,\} \\ &= \{\, \mathbb{v} \mid \Pi^{\top}\mathbb{v} \in \mathfrak{E} \,\} = \mathbb{V}. \end{aligned} \tag{3.20}$$

Since $\Pi\,\mathfrak{S}$ is a closed convex cone, $\mathbb{V}^o = (\Pi\,\mathfrak{S})^{oo} = \Pi\,\mathfrak{S}$ □

Thus the placement is rigid if and only if

$$\Pi\,\mathfrak{S} = \mathscr{R}^o = \mathscr{R}^{\perp}; \tag{3.21}$$

the last holds since $\mathscr{R}$ is a subspace. We re-express this as

Proposition 3.18. *A placement is rigid if and only if*

$$Span\,\{\boldsymbol{\pi}_e \,|\, e \in \mathscr{E}\,\} = \mathscr{R}^{\perp} \tag{3.22}$$

We look at some special placements in which stability and rigidity are equivalent. Following Asimow and Roth ([3]) and Roth and Whiteley ([50]), we call $\mathbb{p}$ a **regular placement** if $\mathbb{q} = \mathbb{p}$ yields a *local maximum* of

$$\dim\,(\,\mathrm{Span}\,\{\,\boldsymbol{\pi}_e(\mathbb{q})\,|\,e \in \mathscr{E}\}) = \mathrm{rank}\,(\Pi(\mathbb{q})); \quad \mathbb{q} \in \mathbb{R}^{3\mathscr{N}} \tag{3.23}$$

More specially, if $\mathbb{q} = \mathbb{p}$ yields a *local maximum* of

$$\dim\,(\,\mathrm{Span}\;\{\,\boldsymbol{\pi}_e(\mathbb{q})\,|\,e \in \mathscr{A}\})\,; \quad \mathbb{q} \in \mathbb{R}^{3\mathscr{N}} \tag{3.24}$$

for all $\mathscr{A} \subseteq \mathscr{E}$, we say that $\mathbb{p}$ is a **general placement**.

From Prop 3.18, recognizing that the range of Π is $\mathscr{R}$, immediately

Lemma 3.19. *If a placement is rigid then the placement is regular.*

Theorem 3.20 (Asimow & Roth)**.** *A placement of a bar structure is rigid if and only if it is stable and the placement is regular.*

Proof. We already have observed that rigidity implies stability. With the lemma, this give us the forward implication.

For the converse, let us assume the placement is not rigid; we will show that it cannot be stable. We need the following concept[2]

Lemma 3.21. *The complete (bar) graph, K formed from the node-set $\mathscr{N}$ of a regular placement is rigid.*

The affine span of the node-set either is a plane or is all of $\mathbb{R}^3$, since we have excluded colinearity. Suppose that is all of space. Then we choose four nodes $\mathbb{p}_0$, and $\mathbb{p}_0 + \mathbf{e}_i$, $i = 1, 2, 3$, where the $\mathbf{e}_i$ are spanning. Given an admissible velocity $\mathbb{v}$, we define a linear map W by $W\mathbf{e}_i = \mathbb{v}_i - \mathbb{v}_0$, using the obvious abbreviation for the velocities of the selected nodes. Since all line segments are not shortened by the velocity, we calculate that

$$W\mathbf{e}_i \cdot \mathbf{e}_i = 0, \quad \text{and}$$
$$W(\mathbf{e}_i - \mathbf{e}_j) \cdot (\mathbf{e}_i - \mathbf{e}_j) = 0$$

[2]In fact, Asimow and Roth define the concept of rigidity using this implicit comparison of the admissible velocities for K and the structure.

so that

$$W\mathbf{e}_i \cdot \mathbf{e}_j = -\mathbf{e}_i \cdot W\mathbf{e}_j \tag{3.25}$$

for all choices of i, j. Thus W is skew and we have the claimed relation for the distinguished nodes.

For any other node, say $\mathbb{p}_\kappa = \mathbb{p}_0 + \mathbf{v}$, we note that $\mathbf{v} \cdot (\mathbb{v}_\kappa - \mathbb{v}_0) = 0$ and $(\mathbf{v} - \mathbf{e}_i) \cdot (\mathbb{v}_\kappa - \mathbb{v}_i) = 0$ for $i = 1, 2, 3$ in order to ensure that the connecting edges all are unchanged in length. Then for each i

$$\begin{aligned}(\mathbb{v}_\kappa - \mathbb{v}_0 - W\mathbf{v}) \cdot \mathbf{e}_i &= (\mathbb{v}_\kappa - \mathbb{v}_0 - W\mathbf{v}) \cdot (\mathbf{e}_i - \mathbf{v}) \\ &= (\mathbb{v}_\kappa - \mathbb{v}_i) \cdot (\mathbf{e}_i - \mathbf{v}) + (W\mathbf{e}_i - W\mathbf{v}) \cdot (\mathbf{e}_i - \mathbf{v}) = 0,\end{aligned} \tag{3.26}$$

which implies that $\mathbb{v}_\kappa = W\mathbf{v} + \mathbb{v}_0$. Hence the velocity is rigid.

If the points are planar, then we have only three nodes in the spanning set, but we may use the condition that the nodes remain in a plane to ensure unique specification of W and the representation of the velocity. (If $\mathbf{n}$ is the normal to the plane, this condition is $(\mathbb{p}_\kappa - \mathbb{p}_0) \cdot W\mathbf{n} + (\mathbb{v}_\kappa - \mathbb{v}_0) \cdot \mathbf{n} = \mathbf{0}$ for all nodes.) We define W by the previous calculations for $\mathbf{e}_1$, $\mathbf{e}_2$ and by

$$W\mathbf{n} = -\mathbf{n} \cdot (\mathbb{v}_1 - \mathbb{v}_0)\,\widehat{\mathbf{e}_1} - \mathbf{n} \cdot (\mathbb{v}_2 - \mathbb{v}_0)\,\widehat{\mathbf{e}_2} \tag{3.27}$$

where $(\widehat{\mathbf{e}_1}, \widehat{\mathbf{e}_2})$ is dual to $(\mathbf{e}_1, \mathbf{e}_2)$ in the subspace. We then use the planar calculation for other nodes along with the previous computations to verify the affine representation. This completes the proof of the lemma.

Assuming the placement is not rigid, we can find a velocity $\mathbb{v}$ not in $\mathscr{R}$ which is admissible for the structure. Since it is *not* admissible for K there are nodes α, β, necessarily *not* ends of an edge of the structure, for which

$$\mathbb{v} \cdot (\mathbb{p}_\alpha - \mathbb{p}_\beta) \neq 0. \tag{3.28}$$

Next, consider the collection of edge vectors $(\boldsymbol{\pi}_e(\mathbb{p}) \mid e \in \mathscr{E})$. Choose a linearly independent subset with the same span; say $(\boldsymbol{\pi}_i(\mathbb{p}) \mid i \in I)$. This cannot span $\mathbb{R}^{3\mathcal{N}}$, since there is a vector, $\mathbb{v}$, orthogonal to all. Since the edge vectors are continuous functions of the placement, this subset remains linearly independent in a neighborhood of $\mathbb{p}$, and because $\mathbb{p}$ is regular, the other edge vectors remain dependent upon these in a neighborhood. Note that even though $\alpha\beta$ is not an edge, we can define $\boldsymbol{\pi}_{\alpha\beta}$; this vector is not in the span of $(\boldsymbol{\pi}_i(\mathbb{p}) \mid i \in I)$, because of (3.28). Again, by continuity, the collection $(\boldsymbol{\pi}_i)$ together with $\boldsymbol{\pi}_{\alpha\beta}$ stays linearly independent in a neighborhood.

Now consider the set of differential equations

$$\boldsymbol{\pi}_i(\mathbb{q}(t)) \cdot \dot{\mathbb{q}}(t) = 0 \quad \forall i; \tag{3.29}$$

$$\boldsymbol{\pi}_{\alpha\beta}(\mathbb{q}(t)) \cdot \dot{\mathbb{q}}(t) = 1. \tag{3.30}$$

There is a half-neighborhood of $t = 0$ in which this has a solution. Because the other edge vectors locally are linear combinations of the $(\boldsymbol{\pi}_i)$, it follows that no edges change in length, while the nodes α, β move further apart, so the motion is not isometric. □

Roth and Whiteley in [50] strengthen the hypothesis to extend this result to tensegrity structures:

Theorem 3.22 (Roth & Whiteley)**.** *Suppose that* $\mathbb{p}$ *is a general placement. Then the tensegrity structure is rigid at* $\mathbb{p}$ *if and only if it is stable there.*

Proof. Rigidity implies stability, as we have noted. Suppose that the placement is not rigid. Consider the set $\widehat{\mathbb{V}}$ of admissible, non-rigid velocities. If all are isometric flexes, then the placement also is not rigid as a bar structure. But the placement is regular, so the previous results says it would be unstable as a bar structure and hence unstable as a tensegrity structure.

Suppose, then, that there is a $\mathbb{v} \in \widehat{\mathbb{V}}$ which tends to shorten at least one edge. We let

$$\mathcal{A} = \{ e \in \mathcal{E} \mid \boldsymbol{\pi}_e \cdot \mathbb{w} = 0 \ \ \forall \mathbb{w} \in \widehat{\mathbb{V}} \} \tag{3.31}$$

Since $\mathbb{v}$ shortens one element, $\mathcal{A} \neq \mathcal{C}$; it is not empty, as we assume there are bars in the structure. As in the previous proof, choose a linearly independent spanning set$(\boldsymbol{\pi}_i)$ from $\{ \boldsymbol{\pi}_e \mid e \in \mathcal{A} \}$.

We cannot use the technique of the last proof, as there may be several edges shortened by $\mathbb{v}$.

Consider the set on which this selected set of edges have fixed lengths:

$$\{ \mathbb{q} \mid \boldsymbol{\lambda}_i(\mathbb{q}) = \boldsymbol{\lambda}_i(\mathbb{p}) \ \ \forall i \}. \tag{3.32}$$

This is a manifold near $\mathbb{p}$ with tangent space at $\mathbb{q}$ normal to all $\boldsymbol{\pi}_i(\mathbb{q})$. It follows that the tangent space at $\mathbb{p}$ also is normal to all edge vectors $\boldsymbol{\pi}_e$ with $e \in \mathcal{A}$. But this also must be true in a neighborhood of $\mathbb{p}$. For if in each neighborhood of $\mathbb{p}$ there were a $\mathbb{q}$ and an $e_o \in \mathcal{A}$ with $\boldsymbol{\pi}_{e_o}(\mathbb{q})$ not normal to the tangent space then the collection of edge vectors from $\mathcal{A}$ would be of higher dimension at $\mathbb{q}$ than at $\mathbb{p}$, contradicting that $\mathbb{p}$ is a general placement. It follows that all edges in $\mathcal{A}$ are of constant length in a neighborhood of $\mathbb{p}$ on the manifold.

We choose a path in the manifold starting at $\mathbb{p}$ whose initial tangent vector is $\mathbb{v}$. All of the edges in $\mathscr{A}$ remain constant on this path, and since $\mathbb{v}$ shortens at least one edge, the same remains true, by continuity, in some neighborhood. Thus we have an admissible path, and the placement is unstable. □

These results are useful, but leave much to be done; as we shall see, placements which are stable but not rigid are of great interest, and these cannot be general placements. We formalize this:

Corollary 3.23. *If a placement $\mathbb{p}$ of a tensegrity structure is stable but not rigid then it cannot be general, ie, there is a set of edges $\mathscr{A}$ such that in any neighborhood of $\mathbb{p}$*

$$\mathbb{q} \mapsto dim\ (\ Span\,\{\ \boldsymbol{\pi}_e(\mathbb{q})\,|\,e \in \mathscr{A}\})$$

has values greater or equal that at $\mathbb{q} = \mathbb{p}$.

3.3 Stability and Expansions of Motions

The concept of second-order stability was introduced by Connelly and Whiteley [13]. To motivate it, consider a motion $\mathbb{q}(t)$ away from $\mathbb{p}$, and take the first and second derivatives of the associated length function:

$$\lambda_e(0) = B_e\mathbb{p} \cdot \mathbb{p} \tag{3.33a}$$

$$\dot{\lambda}_e(0) = 2B_e\mathbb{p} \cdot \dot{\mathbb{q}}(0), \tag{3.33b}$$

$$\ddot{\lambda}_e(0) = 2B_e\dot{\mathbb{q}}(0) \cdot \dot{\mathbb{q}}(0) + 2B_e\mathbb{p} \cdot \ddot{\mathbb{q}}(0). \tag{3.33c}$$

Accordingly, we call a pair $(\mathbb{v}, \mathbb{a}) \in \mathbb{R}^{3\mathcal{N}} \times \mathbb{R}^{3\mathcal{N}}$ an **admissible velocity-acceleration** pair for $\mathbb{p}$ if they satisfy

$$B_b\mathbb{p} \cdot \mathbb{v} = 0, \text{ and } \quad B_b\mathbb{v} \cdot \mathbb{v} + B_b\mathbb{p} \cdot \mathbb{a} = 0 \tag{3.34a}$$

for each bar $b \in \mathscr{B}$, and *either*

$$B_c\mathbb{p} \cdot \mathbb{v} < 0 \tag{3.34b}$$

or

$$B_c\mathbb{p} \cdot \mathbb{v} = 0, \text{ and } \quad B_c\mathbb{v} \cdot \mathbb{v} + B_c\mathbb{p} \cdot \mathbb{a} \le 0 \tag{3.34c}$$

for each cable c. Of course, rigid motions give rise to admissible velocity-acceleration pairs.

Definition 3.24. *A placement* $\mathbb{p}$ *is* **second-order stable** *if the only admissible velocity-acceleration pairs* $(\mathbb{v}, \mathbb{a})$ *are those with* $\mathbb{v}$ *rigid.*

Clearly a placement which is rigid is second-order stable, but also

Theorem 3.25 (Connelly & Whiteley)**.** *A placement which is second-order stable is stable.*

We defer the proof; it is simplest to prove it in Rep, and we must first establish some equivalences.

Let $\mathbb{p}^* \in$ Rep be rigidly equivalent to $\mathbb{p}$ via $\mathbb{p}^* = Q\,(\mathbb{p} - \mathbb{p}^{\dagger}_{\alpha})$. In (3.12) and (3.13) we have the general form of the velocity and acceleration change under equivalent motions from the two placements. Since the motions from $\mathbb{p}^*$ remain in Rep, it follows that given an admissible velocity-acceleration pair for $\mathbb{p}$ we can choose skew maps W and S to ensure that the velocity-acceleration pair for $\mathbb{p}^*$ are each in $\mathcal{U}$.

Then for the equivalent $\mathbb{p}$, $\mathbb{p}^*$ we relate the factors from the second-order stability test. First,

$$B_e\mathbb{p}^* \cdot \mathbb{v}^* = B_e\mathbb{p} \cdot \mathbb{v},$$

and

$$\begin{aligned}
B_e\mathbb{v}^* \cdot \mathbb{v}^* &= B_e(W\mathbb{p}^* + Q(\mathbb{v} - \mathbb{v}^{\dagger}_{\alpha}) \cdot (W\mathbb{p}^* + Q(\mathbb{v} - \mathbb{v}^{\dagger}_{\alpha}) \\
&= -B_e\mathbb{p}^* \cdot W^2\mathbb{p}^* + 2B_eW\mathbb{p}^* \cdot Q\mathbb{v} + B_e\mathbb{v} \cdot \mathbb{v} \\
&= -B_e\mathbb{p}^* \cdot W^2\mathbb{p}^* - 2B_e\mathbb{p}^* \cdot WQ\mathbb{v} + B_e\mathbb{v} \cdot \mathbb{v}, \qquad (3.35)
\end{aligned}$$

while

$$\begin{aligned}
B_e\mathbb{p}^* \cdot \mathbb{a}^* &= B_e\mathbb{p}^* \cdot \left[(S + W^2)Q(\mathbb{p} - \mathbb{p}^{\dagger}_{\alpha}) + 2WQ(\mathbb{v} - \mathbb{v}^{\dagger}_{\alpha}) + Q(\mathbb{a} - \mathbb{a}^{\dagger}_{\alpha})\right] \\
&= B_e\mathbb{p}^* \cdot S\mathbb{p}^* + B_e\mathbb{p}^* \cdot W^2\mathbb{p}^* + 2B_e\mathbb{p}^* \cdot WQ\mathbb{v} + B_e\mathbb{p} \cdot \mathbb{a} \\
&= B_e\mathbb{p}^* \cdot W^2\mathbb{p}^* + 2B_e\mathbb{p}^* \cdot WQ\mathbb{v} + B_e\mathbb{p} \cdot \mathbb{a}. \qquad (3.36)
\end{aligned}$$

Then we combine these to find

$$B_e\mathbb{p}^* \cdot \mathbb{v}^* = B_e\mathbb{p} \cdot \mathbb{v} \qquad (3.37)$$

and

$$B_e\mathbb{v}^* \cdot \mathbb{v}^* + B_e\mathbb{p}^* \cdot \mathbb{a}^* = B_e\mathbb{v} \cdot \mathbb{v} + B_e\mathbb{p} \cdot \mathbb{a} \qquad (3.38)$$

Proposition 3.26. *If $\mathbb{p}^* \in$ Rep is rigidly equivalent to $\mathbb{p}$ with rotation Q, and if velocities and accelerations are related by (3.12) and (3.13) then*

$$B_e\mathbb{p}^* \cdot \mathbb{v}^* = B_e\mathbb{p} \cdot \mathbb{v}$$

and

$$B_e\mathbb{v}^* \cdot \mathbb{v}^* + B_e\mathbb{p}^* \cdot \mathbb{a}^* = B_e\mathbb{v} \cdot \mathbb{v} + B_e\mathbb{p} \cdot \mathbb{a}.$$

Hence $\mathbb{p}$ is second-order stable if and only if $\mathbb{p}^*$ is.

To complete the proof we need only note that the mapping of velocity acceleration into Rep carries a rigid velocity into a rigid velocity and conversely a rigid velocity for $\mathbb{p}^*$ has as preimage only rigid velocities.

Finally, by considering projections into $\mathcal{U}$, it is easy to see that we may restate the criterion for Rep:

Corollary 3.27. *A placement in Rep is second-order stable if and only if any admissible velocity-acceleration pair in $\mathcal{U}$ has zero velocity.*

In [1] Alexandrov, dealing only with bar structures, generalizes this expansion of the motion, and provides some more sufficient conditions for stability. We follow his computations. For convenience, let us work in Rep, with **velocity subspace** $\mathcal{U}$.

A motion from $\mathbb{p}^*$ is written as

$$\mathbb{q}(t) = \sum_{s=0}^{\infty} t^s \mathbb{q}_s \qquad \mathbb{q}_0 = \mathbb{p}^*. \tag{3.39}$$

All of the coefficients except $\mathbb{q}_0$ lie in $\mathcal{U}$.

For each edge e, we calculate

$$\lambda_e(t) = \lambda_e(\mathbb{p}) + \frac{1}{2}\sum_{s=1}^{\infty}\left(\sum_{r=0}^{s} B_e\, \mathbb{q}_r \cdot \mathbb{q}_{s-r}\right) t^s. \tag{3.40}$$

First, consider a bar b. To be admissible the motion must satisfy

$$\sum_{r=0}^{s} B_b\, \mathbb{q}_r \cdot \mathbb{q}_{s-r} = 0 \qquad s = 1, 2, \ldots, \tag{3.41}$$

and since $B_b\, \mathbb{q}_0 = \boldsymbol{\pi}_b$ we have the **recursion relation**

$$2\boldsymbol{\pi}_b \cdot \mathbb{q}_s = -\sum_{r=1}^{s-1} B_b\, \mathbb{q}_r \cdot \mathbb{q}_{s-r} \qquad s = 1, 2, \ldots. \tag{3.42}$$

For reference, the first few terms are

$$\begin{aligned} 2\,\boldsymbol{\pi}_b \cdot \mathbb{q}_1 &= 0 \\ 2\,\boldsymbol{\pi}_b \cdot \mathbb{q}_2 &= -B_b\,\mathbb{q}_1 \cdot \mathbb{q}_1 \\ 2\,\boldsymbol{\pi}_b \cdot \mathbb{q}_3 &= -2\,B_b\,\mathbb{q}_2 \cdot \mathbb{q}_1 \\ 2\,\boldsymbol{\pi}_b \cdot \mathbb{q}_4 &= -2\,B_b\,\mathbb{q}_1 \cdot \mathbb{q}_3 - B_b\,\mathbb{q}_2 \cdot \mathbb{q}_2. \end{aligned} \tag{3.43}$$

The first two are part of the criterion that $\mathbb{q}_1,\ 2\mathbb{q}_2$ be a velocity-acceleration pair.

For a cable c the recursion may truncate. The conditions are

$$\sum_{r=0}^{s} B_c\,\mathbb{q}_r \cdot \mathbb{q}_{s-r} \le 0 \qquad p = 1, 2, \ldots, \tag{3.44}$$

or the recursion

$$2\,\boldsymbol{\pi}_c \cdot \mathbb{q}_s \le -\sum_{r=1}^{s-1} B_c\,\mathbb{q}_r \cdot \mathbb{q}_{s-r} \qquad p = 1, 2, \ldots, \tag{3.45}$$

with the understanding that the recursion truncates at the first p for which inequality obtains. Alexandrov's generalization of the second-order stability condition, expanded to tensegrity structures, follows:

Lemma 3.28. *If there is an integer s such that for some $b \in \mathscr{B}$ (3.42) has no solution in $\mathcal{U}$, or if there is an integer s such that for some $c \in \mathscr{C}$ not already shortened (3.45) has no solution in $\mathcal{U}$, then the placement is stable.*

We wish to prove the second-order stability condition (Theorem 3.25); we need to consider the equivalent statement in Rep. If a motion $\mathbb{q}(\cdot)$ yielding initial velocity and acceleration $\mathbb{v}$ and $\mathbb{a}$ is equivalent to the motion $\mathbb{q}^*(\cdot)$ in Rep, the initial velocity and acceleration of the latter both lie in $\mathcal{U}$ and have the forms

$$\begin{aligned} \mathbb{v}^* &= WQ(\mathbb{p} - \mathbb{p}_\alpha^\dagger) + Q(\mathbb{v} - \mathbb{v}_\alpha^\dagger), \\ \mathbb{a}^* &= (S + W^2)Q(\mathbb{p} - \mathbb{p}_\alpha^\dagger) + 2WQ(\mathbb{v} - \mathbb{v}_\alpha^\dagger) + Q(\mathbb{a} - \mathbb{a}_\alpha^\dagger), \end{aligned} \tag{3.46}$$

where both S and W are skew and Q is the rotation map in the rigid map of $\mathbb{p}$ into $\mathbb{p}^*$. We can reuse previous arguments to show that given any $\mathbb{v}$, $\mathbb{a}$ we can find W, S to ensure that the resulting $\mathbb{v}^*$, $\mathbb{a}^*$ are in $\mathcal{U}$. Note that (3.46) is an injection between $(\mathbb{v}, \mathbb{a})$ and $(\mathbb{v}^*, \mathbb{a}^*)$.

Proof. Suppose that there is an admissible motion from $\mathbb{p}^* \in$ Rep. Consider its expansion as in (3.39).

If $\mathbb{q}_1 \neq 0$, then for each bar b and cable c

$$\boldsymbol{\pi}_b \cdot \mathbb{q}_1 = 0 \;\text{ and }\; 2\boldsymbol{\pi}_b \cdot \mathbb{q}_2 + B_b \mathbb{q}_1 \cdot \mathbb{q}_1 = 0 \tag{3.47}$$

and

$$\begin{aligned}&\boldsymbol{\pi}_c \cdot \mathbb{q}_1 < 0 \;\text{ or}\\ &\boldsymbol{\pi}_c \cdot \mathbb{q}_1 = 0 \;\text{ and }\; 2\boldsymbol{\pi}_c \cdot \mathbb{q}_2 + B_c \mathbb{q}_1 \cdot \mathbb{q}_1 \leq 0\end{aligned}$$

so that $\mathbb{q}_1, \frac{1}{2}\mathbb{q}_2$ form an admissible velocity-acceleration pair.

If $\mathbb{q}_1 = 0$, let $\mathbb{q}_r$ be the first non-zero coefficient. From the recursion relation, for each bar b and cable c

$$\boldsymbol{\pi}_b \cdot \mathbb{q}_r = 0 \;\text{ and }\; \boldsymbol{\pi}_c \cdot \mathbb{q}_r \leq 0.$$

Consider the recursion rule for $\mathbb{q}_{2r}$:

$$2\boldsymbol{\pi}_b \cdot \mathbb{q}_{2r} + B_b \mathbb{q}_r \cdot \mathbb{q}_r = 0,$$

and, if $\mathbb{q}_r < 0$,

$$2\boldsymbol{\pi}_c \cdot \mathbb{q}_{2r} + B_c \mathbb{q}_r \cdot \mathbb{q}_r \leq 0.$$

Hence $\mathbb{q}_r, \mathbb{q}_{2r}$ form an admissible velocity-acceleration pair.

Hence if there is an admissible motion then there is an admissible velocity-acceleration pair. Thus non-existence of such a pair implies stability. □

4 Forces and Stresses

Now we introduce the concept of forces into our calculations.

4.1 Resolvable Forces

We represent a set of **externally imposed forces** applied to the structure as a vector of nodal forces:

$$\mathbb{f} \in \mathbb{R}^{3\mathcal{N}}. \tag{4.1}$$

We are interested in equilibrium of structures, so *we consider only those sets of applied forces whose net force and moment is zero.* It is convenient to express this by saying that if $\mathbb{p}$ is a given placement of the structure, then a force $\mathbb{f}$ is **equilibrated** relative $\mathbb{p}$ if

$$\mathbb{f} \in \mathscr{R}(\mathbb{p})^{\perp}. \tag{4.2}$$

Each edge of the structure may carry a force; we measure their magnitude in a convenient way by a **stress vector**

$$\boldsymbol{\omega} \in \mathbb{R}^{\mathscr{E}}. \tag{4.3}$$

ω_e is the force carried by the edge e divided by the length of e. It is taken to be positive if the force is a tension, so that it generates a force on the node e_α as

$$-\boldsymbol{\omega}_e \left(\boldsymbol{\pi}_{e_\alpha} - \boldsymbol{\pi}_{e_\omega}\right). \tag{4.4}$$

The totality of the edge-forces acting on a node is then the corresponding entry in the sum

$$-\sum_e \boldsymbol{\omega}_e \boldsymbol{\pi}_e \tag{4.5}$$

and **balance of external and edge-forces** is expressed as

$$\mathbb{f} = \sum_e \boldsymbol{\omega}_e \boldsymbol{\pi}_e. \tag{4.6}$$

or

$$\mathbb{f} = \Pi \boldsymbol{\omega} \tag{4.7}$$

By (4.7) and the corresponding form (2.31) for edge-strains, we have the **equation of virtual work**: for any ω and any $\mathbb{v}$

$$\mathbb{f} \cdot \mathbb{v} = \boldsymbol{\omega} \cdot \boldsymbol{\epsilon}, \tag{4.8}$$

outlining the duality between the forces-velocities and stress-stretching pairs. A further characterization of virtual work is allowed by the continued calculation

$$\begin{aligned} \mathbb{f} \cdot \mathbb{v} = \Pi\, \boldsymbol{\omega} \cdot \mathbb{v} = \sum_e \boldsymbol{\omega}_e\, B_e \mathbb{p} \cdot \mathbb{v} = \\ = \left(\sum_e \boldsymbol{\omega}_e\, B_e \right) \mathbb{p} \cdot \mathbb{v} = \Omega\, \mathbb{p} \cdot \mathbb{v} \end{aligned} \tag{4.9}$$

where we have introduced the (symmetric) **stress operator**

$$\Omega = \sum \omega_e B_e \tag{4.10}$$

of Connelly [10, 13]. Note that Ω is determined only by $\boldsymbol{\omega}$ and the graph of the structure, *ie,* it is independent of placement.

Recognizing that cables can carry only tension, we call a stress a **proper stress** if

$$\boldsymbol{\omega} \in \mathfrak{S}, \tag{4.11}$$

(recall the definition of the stress cone as $\mathfrak{S} = \{ \boldsymbol{\mu} \in \mathbb{R}^{\mathscr{E}} \mid \boldsymbol{\mu}_c \geq 0 \;\; \forall c \in \mathscr{C} \}$) and call the stress **strict** if $\omega_c > 0$ for all cables c.

An external force $\mathbb{f}$ balanced by a stress vector is called **resolvable**. By (4.6) and (3.17), we have

Proposition 4.1. *Any resolvable external force is equilibrated.*

Of particular interest is the cone of **properly resolvable forces** for $\mathbb{p}$

$$\mathbb{F}(\mathbb{p}) = \Pi\, \mathfrak{S} \subseteq \mathscr{R}^{\perp}. \tag{4.12}$$

We already have used this set in Lemma 3.17 and in Proposition 3.18. We repeat these in

Proposition 4.2. *For any placement the convex cone of properly resolvable forces $\mathbb{F}$ is the polar of the cone of admissible velocities, and the placement is rigid if and only if $\mathbb{F}$ is a linear space.*

The classical definition of mechanical rigidity was formulated in [33]

Definition 4.3 (Maxwell)**.** *A placement of* **T** *is* **statically rigid** *if each equilibrated force is properly resolvable.*

The following was proved for bar-trusses by Maxwell, and generalized to tensegrity structures in [50]

Theorem 4.4 (Maxwell, Roth & Whiteley)**.** *Rigidity is equivalent to statical rigidity.*

Proof. Rigidity means that $\mathbb{V} = \mathscr{R}$; statically rigidity means that $\mathbb{F} = \mathscr{R}^{\perp}$. But Lemma 3.17 ensures that the two statements are equivalent. ◻

Corollary 4.5. *A structure in a given placement can support all applied loads if and only if it is not flexible.*

4.2 Prestresses and Stability; the Second-order Stress Test

The notion of prestresses turns out to be of central importance in the development of the theory, as experiments with tensegrity constructs demonstrate.

A stress vector ω is a **prestress** if it resolves $\mathbf{0}$, *i.e.*, if

$$\sum_e \omega_e \boldsymbol{\pi}_e = \mathbf{0} \tag{4.13}$$

or

$$\Pi\omega = \mathbf{0} \tag{4.14}$$

Thus a prestress represents a linear dependency amongst the set $\{\boldsymbol{\pi}_e\}$ of edge vectors or, equivalently, is in the null space of Π. Both points of view are useful in what follows. We may call $(\mathbb{p}, \omega)$ a **tensegrity pair** if ω is a prestress for the placement $\mathbb{p}$

The equation of virtual work (4.8) leads to a useful computation, which says that a prestress does no work under any velocity applied to the system.

Lemma 4.6. *If ω is a prestress for the placement then for any velocity field $\mathbb{v}$,*

$$\mathbb{f} \cdot \mathbb{v} = \boldsymbol{\omega} \cdot \boldsymbol{\epsilon} = \sum_e \omega_e \boldsymbol{\epsilon}_e = 0. \tag{4.15}$$

If ω is proper and $\mathbb{v}$ is admissible, then for each edge e

$$\omega_e \boldsymbol{\epsilon}_e = 0. \tag{4.16}$$

Proof. The first statement is immediate. The second is a consequence, since for each bar we have $\boldsymbol{\epsilon}_b = 0$, while for each cable $\omega_c \boldsymbol{\epsilon}_c \leq 0$ by the sign restrictions on proper prestresses and admissible velocities. □

Some technical results will be useful later.

Proposition 4.7. *Given a stress vector $\omega \in \mathbb{R}^{\mathscr{E}}$*

1. *ω is a prestress for the placement $\mathbb{p}$ iff the corresponding stress operator has the property*
$$\Omega\, \mathbb{p} = 0. \tag{4.17}$$

2. *If ω is a prestress for the placement $\mathbb{p}$ then*
$$\omega \cdot \boldsymbol{\lambda}(\mathbb{p}) = 0. \tag{4.18}$$

3. *If ω is a prestress for the placement $\mathbb{p}$ then it is a prestress for any placement $\mathbb{q}$ which is an affine image of $\mathbb{p}$. In particular, it is a prestress for all placements equivalent to $\mathbb{p}$.*

Proof. The first is just a restatement of $\sum \boldsymbol{\omega}_e B_e \mathbb{p} = 0$. To obtain the second, we take the dot product of this expression with $\mathbb{p}$. The last result is an immediate consequence of the fact that linear maps filter through B_e (equation (2.26)). $\square$

Now we can relate stability and prestresses. An interesting relation between structures and the equivalent bar-truss is given in [50]:

Theorem 4.8 (Roth & Whiteley)**.** *A placement is rigid as a tensegrity structure if and only if it is is rigid as a bar-truss and the placement admits a strict proper prestress.*

Proof. If $\mathbb{F}$ denotes the set of resolvable forces for the tensegrity structure the corresponding set of forces for the bar-truss, $\mathbb{F}_s$ is the range of Π:

$$\mathbb{F}_s = \Pi\, \mathbb{R}^{\mathscr{E}} = \operatorname{Span} \mathbb{F}. \tag{4.19}$$

Generally $\mathbb{F} \subseteq \mathbb{F}_s$ They are equal exactly when $-\boldsymbol{\pi}_c \in \mathbb{F}$ for each cable c, and hence exactly when $\mathbb{F}$ is a subspace. We use the lemma

Lemma 4.9. *There exists a strict prestress for the placement if and only if $\mathbb{F}$ is a subspace.*

Proof. If $\boldsymbol{\omega}$ is a strict proper prestress we have

$$\sum \boldsymbol{\omega}_b \boldsymbol{\pi}_b + \sum \boldsymbol{\omega}_c \boldsymbol{\pi}_c = \mathbf{0} \tag{4.20}$$

with all $\boldsymbol{\omega}_c > 0$. This means that for any cable c_o,

$$-\boldsymbol{\pi}_{c_o} = \frac{1}{\boldsymbol{\omega}_{c_o}} \left(\sum \boldsymbol{\omega}_b \pi_b + \sum_{c \neq c_o} \boldsymbol{\omega}_c \pi_c \right), \tag{4.21}$$

and, as noted above, this ensures that $\mathbb{F}$ is a subspace.

Conversely, if $\mathbb{F}$ is a subspace, then for any $c_o \in \mathscr{C}$ we have $-\boldsymbol{\pi}_{c_o} \in \mathbb{F}$, so we can find coefficients such that

$$-\boldsymbol{\pi}_{c_o} = \sum \boldsymbol{\epsilon}_b \pi_b + \sum_c \boldsymbol{\epsilon}_c \pi_c, \quad \boldsymbol{\epsilon}_c \geq 0. \tag{4.22}$$

Thus there is a strict proper prestress (with the stress in c_o being $1 + \boldsymbol{\epsilon}_{c_o}$). $\square$

If the tensegrity structure is rigid then so is the bar structure and the lemma ensures that there is a strict proper prestress (since $\mathbb{F} = \mathscr{R}^\top$).

If the bar structure is rigid then $\mathbb{F}_s = \mathscr{R}^\top$, and if there is a strict proper prestress then $\mathbb{F}$ is a subspace and $\mathbb{F} = \mathbb{F}_s = \mathscr{R}^\top$. □

Corollary 4.10. *There exists a strict proper prestress if and only if $\mathbb{V}$ is a subspace.*

The connection with $\mathbb{V}$ follows since $\mathbb{F}^o = \mathbb{V}$.

Thus we have a connection of prestress and rigidity. We wish to examine the connection to stability. The following result from [50], *cf.* [64], is a first step.

Theorem 4.11 (Roth & Whiteley)**.** *Given a placement and $c_o \in \mathscr{C}$, there is an admissible velocity with stretching $\epsilon_{c_o} < 0$ if and only if every proper prestress has $\omega_{c_o} = 0$.*

In fact this is a consequence of the next theorem, but it has a direct proof which is more intuitive than that one.

Proof. Consider the velocity $\mathbb{v}$. By Lemma 4.6, since the velocity is admissible, for any proper prestress $\boldsymbol{\omega}$ we have

$$\omega_c \epsilon_c = 0$$

for each cable c. But $\epsilon_{c_o} < 0$, which means that $\omega_{c_o} = 0$.

Conversely, if every proper prestress has $\boldsymbol{\omega}_{c_o} = 0$ it follows from the argument in Lemma 4.9 that

$$-\pi_{c_o} \notin \mathbb{F}. \tag{4.23}$$

But this ensures that there is a $\mathbb{v} \in \mathbb{V} = \mathbb{F}^0$ such that

$$-\pi_{c_o} \cdot \mathbb{v} > 0, \tag{4.24}$$

ie $\boldsymbol{\epsilon}_{c_o} < 0$, while

$$\gamma \cdot \mathbb{v} \leq 0 \tag{4.25}$$

for all vectors γ in $\mathbb{F}$. In fact it must be true that $\gamma \cdot \boldsymbol{\pi}_b = 0$ for all bars b, since $-\boldsymbol{\pi}_b \in \mathbb{F}$. Hence $\mathbb{v}$ is admissible. □

It is then a simple step to generalize this to

Corollary 4.12. *Given a placement of* $\mathbf{T}$ *and* $\mathscr{A} \subset \mathscr{C}$*, there is an admissible velocity with* $\boldsymbol{\epsilon}_c < 0$ *for all* $c \in \mathscr{A}$ *if and only if every proper prestress has* $\omega_c = 0$ *for all* $c \in \mathscr{A}$*.*

This result appears stated in a slightly different form as Prop 5.1.1 in [13]. Their Prop 5.1.2 is the following, which can be regarded as a generalization of the virtual work statement (4.15), or alternatively, a convexification of the statement of orthogonality of the range of the operator and the null-space of its transpose in linear algebra.

Proposition 4.13. *Given a placement and given* $\boldsymbol{\epsilon} \in \mathbb{R}^{\mathscr{E}}$ *there exists a velocity* $\mathbb{w}$ *such that*

$$\boldsymbol{\pi}_b \cdot \mathbb{w} = \boldsymbol{\epsilon}_b \text{ for every } b \in \mathscr{B}, \tag{4.26a}$$

$$\boldsymbol{\pi}_c \cdot \mathbb{w} \leq \boldsymbol{\epsilon}_c \text{ for every } c \in \mathscr{C}, \tag{4.26b}$$

if and only if for every proper prestress $\boldsymbol{\omega}$

$$\boldsymbol{\omega} \cdot \boldsymbol{\epsilon} \geq 0. \tag{4.27}$$

Proof. Let $\mathscr{H}$ denote the range of $\Pi^\top$. Then (4.26) is the statement that

$$\boldsymbol{\epsilon} \in \mathscr{H} - \mathfrak{E}. \tag{4.28}$$

$-\mathfrak{E}$ is a closed convex cone, so that $\mathscr{H} - \mathfrak{E}$ also is one. For any closed convex cone $\mathscr{D}$ we have $\mathscr{D}^{oo} = \mathscr{D}$, so that (4.28) is equivalent to

$$\boldsymbol{\epsilon} \in (\mathscr{H} - \mathfrak{E})^{oo}. \tag{4.29}$$

Now we need a computation for polars of such sets.

Lemma 4.14. *For any subspace* $\mathscr{U}$ *and any convex closed cone* $\mathscr{D}$

$$(\mathscr{U} + \mathscr{D})^o = \mathscr{U}^\perp \cap \mathscr{D}^o. \tag{4.30}$$

Proof. If $\mathbf{a} \in \mathscr{U}^\perp \cap \mathscr{D}^o$ then for all $\mathbf{u} \in \mathscr{U}$, $\mathbf{d} \in \mathscr{D}$ we have $\mathbf{a} \cdot (\mathbf{u} + \mathbf{d}) \leq 0$. Conversely if $\mathbf{a} \cdot (\mathbf{u} + \mathbf{d}) \leq 0$ for all such vectors then also $\mathbf{a} \cdot (-\mathbf{u} + \mathbf{d}) \leq 0$ and it follows that $\mathbf{a} \cdot \mathbf{u} = 0$, $\mathbf{a} \cdot \mathbf{d} \leq 0$. □

Hence (4.29) becomes

$$\boldsymbol{\epsilon} \in (\mathscr{H}^\perp \cap (-\mathfrak{S}))^o = -(\mathscr{H}^\perp \cap \mathfrak{S})^o. \tag{4.31}$$

But $\mathscr{H}^\perp$ is the null space of Π, *ie*, the set of null stresses, so that $\mathfrak{S} \cap \mathscr{H}^\perp$ is the set of proper prestresses. Thus (4.31) says that $\boldsymbol{\epsilon} \cdot (-\omega) \leq 0$ for any proper prestress, which is the same as (4.27). □

An immediate application of Proposition 4.13 gives an important test for the existence of velocity-acceleration pairs.

Corollary 4.15. *An admissible velocity* $\mathbb{v}$ *extends to an admissible velocity-acceleration pair if and only if for every proper prestress* $\boldsymbol{\omega}$

$$\sum \omega_e \mathbb{v} \cdot B_e \mathbb{v} = \mathbb{v} \cdot \Omega\, \mathbb{v} \leq 0. \tag{4.32}$$

Proof. From (3.34) we see that given an admissible $\mathbb{v}$ we seek an $\mathbb{a}$ such that

$$\boldsymbol{\pi}_b \cdot \mathbb{a} = -\mathbb{v} \cdot B_b \mathbb{v} \tag{4.33a}$$

$$\boldsymbol{\pi}_c \cdot \mathbb{a} \leq -\mathbb{v} \cdot B_c \mathbb{v} \tag{4.33b}$$

for all bars and those cables for which $B_c \mathbb{p} \cdot \mathbb{v} = 0$. For the set $\mathscr{A}$ of cables which have $B_c \mathbb{p} \cdot \mathbb{v} < 0$ there is no restriction; for these let us replace (4.33b) by

$$\boldsymbol{\pi}_c \cdot \mathbb{a} \leq -1. \tag{4.33c}$$

The proposition says that (4.33) can be true if and only if

$$-\sum_{\mathscr{E}\setminus\mathscr{A}} \omega_e \mathbb{v} \cdot B_e \mathbb{v} - \sum_{\mathscr{A}} \omega_c \geq 0. \tag{4.34}$$

Since $\omega_c = 0$ for all $c \in \mathscr{A}$, this is equivalent to

$$-\sum \omega_e \mathbb{v} \cdot B_e \mathbb{v} \geq 0. \tag{4.35}$$

□

Since second-order stability implies stability we have the **second-order stress test** of Connelly and Whiteley [13].

Corollary 4.16. *A sufficient condition for stability of a placement is that for some proper prestress*

$$\mathbb{v} \cdot \Omega\, \mathbb{v} = \mathbb{v} \cdot \sum \omega_e B_e \mathbb{v} > 0, \tag{4.36a}$$

or, equivalently,

$$\sum \omega_e \left(\mathbb{v}_{e_\alpha} - \mathbb{v}_{e_\omega}\right)^2 > 0, \tag{4.36b}$$

for all non-rigid admissible velocities.

An alternative, weaker version of this may be useful.

Corollary 4.17. *A sufficient condition for stability of a placement is that for each non-rigid admissible velocity there is a proper prestress for which*

$$\mathbb{v} \cdot \Omega\, \mathbb{v} = \mathbb{v} \cdot \quad = \mathbb{v} \cdot \sum \omega_e B_e \mathbb{v} > 0, \tag{4.37}$$

Note that for an affine velocity $\mathbb{v} = W\mathbb{p} + \mathbf{v}^\dagger$ we have

$$\sum \omega_e B_e \mathbb{v} = W \sum \omega_e B_e \mathbb{p} = 0\,, \tag{4.38}$$

so that

$$\mathbb{v} \cdot \Omega \mathbb{v} = 0, \tag{4.39}$$

and the second-order stress test fails if there is an admissible non-rigid affine velocity.

It follows that a way of ensuring Proposition 4.16 hold is Connelly's condition of **super-stability** ([15]): a placement is stable if

- There is a strict proper prestress.
- Ω is positive semi-definite.
- The rank of Ω is maximal.
- Aside from rigid velocities no affine velocities are admissible .

These ensure that there are no non-isometric flexes and that Ω is positive-definite on non-rigid velocities.

A physical motivation for the criterion in Proposition 4.17 is given by Calladine and Pellegrino [7]. Consider that

$$\Pi(\mathbb{p}) = [\ldots B_e \mathbb{p} \ldots]\,. \tag{4.40}$$

If the admissible velocity $\mathbb{v}$ is regarded as an infinitesimal perturbation of $\mathbb{p}$ the perturbed geometric matrix is

$$\Pi(\mathbb{p} + \mathbb{v}) = [\ldots (B_e \mathbb{p} + B_e \mathbb{v}) \ldots] = \Pi(\mathbb{p}) + \Pi(\mathbb{v}). \tag{4.41}$$

A prestress for $\mathbb{p}$ is not necessarily one for the new placement; the force required to maintain the given prestress in the new placement would be

$$\mathbb{f} = \Pi(\mathbb{p} + \mathbb{v})\,\omega = \Pi(\mathbb{v})\omega = \sum \omega_e B_e \mathbb{v}, \tag{4.42}$$

and (4.37) is then the statement is

$$\mathbb{f} \cdot \mathbb{v} > 0, \tag{4.43}$$

that *positive work must be done to move the structure from its original placement.*

4.3 Stress in Rep

Rep is a special type of constrained structure. We will discuss general constraints later, but here will verify that the criteria for stability developed in the previous section will apply without change to this case. Rep is an affine subspace (of the form $\mathbb{b} + \mathcal{U}$), as verified in Proposition 3.6; the distinguishing property of Rep is that stated in Proposition 3.7:

$$\mathcal{U} \oplus \mathcal{R}(\mathbb{p}^*) = \mathbb{R}^{3\mathcal{N}}. \tag{4.44}$$

For any structure constrained to an affine space one distinguishes a class of applied forces which do no work on velocities in the associated subspace of velocities. These are called **reaction forces**, and, physically, represent forces needed to maintain the constraints in a motion. For Rep the space of reaction forces is $\mathcal{U}^\perp$, and they have a special property:

Proposition 4.18. *If $\mathbb{p}^* \in$ Rep then given any force $\mathbb{f} \in \mathbb{R}^{3\mathcal{N}}$ there is a reaction force $\mathbb{r}$ such that $\mathbb{f} + \mathbb{r}$ is balanced.*

Proof. By (4.44) we have $\mathcal{U}^\perp \oplus \mathcal{R}(\mathbb{p}^*)^\perp = \mathbb{R}^{3\mathcal{N}}$ and hence we can write $\mathbb{f} = -\mathbb{r} + \mathbb{d}$, with $\mathbb{d} \in \mathcal{R}(\mathbb{p}^*)^\perp$. By definition $\mathbb{d}$ is balanced. □

For a general constrained structure there may be prestresses which resolve purely reactionary forces. Here this cannot be.

Proposition 4.19. *If $\mathbb{p}^* \in$ Rep then for any $\boldsymbol{\omega} \in \mathbb{R}^{\mathcal{E}}$*

$$\Pi\boldsymbol{\omega} \notin \mathcal{U}^\perp. \tag{4.45}$$

This is because the range of Π is in $\mathcal{R}(\mathbb{p}^*)^\perp$, which meets $\mathcal{U}^\perp$ only in $\mathbf{0}$.

Now we consider the specialization of Proposition 4.13. Given an $\boldsymbol{\epsilon} \in \mathbb{R}^{\mathcal{E}}$ we seek a $\mathbb{w} \in \mathcal{U}$ with

$$\boldsymbol{\pi}_b \cdot \mathbb{w} = \boldsymbol{\epsilon}_b \text{ for every } b \in \mathcal{B},$$
$$\boldsymbol{\pi}_c \cdot \mathbb{w} \leq \boldsymbol{\epsilon}_c \text{ for every } c \in \mathcal{C},$$

It is convenient to restate this by allowing $\mathbb{w}$ to be any vector in $\mathbb{R}^{3\mathcal{N}}$ and replacing each $\boldsymbol{\pi}_e$ by $P\boldsymbol{\pi}_e$, where P is the orthogonal projection of $\mathbb{R}^{3\mathcal{N}}$ onto $\mathcal{U}$. This replaces Π in the proposition by $P\Pi$ and the necessary and sufficient condition is replaced by

$$\boldsymbol{\epsilon} \cdot \boldsymbol{\omega} \leq 0 \quad \forall \boldsymbol{\omega} \in \mathfrak{S} \text{ such that } P\Pi\boldsymbol{\omega} = \mathbf{0}. \tag{4.46}$$

But $P\Pi\boldsymbol{\omega} = \mathbf{0}$ means $\Pi\boldsymbol{\omega} \in \mathcal{U}^\perp$ which can be true only if $\Pi\boldsymbol{\omega} = \mathbf{0}$. We restate the result for this case.

Proposition 4.20. *Given a placement in Rep and given $\boldsymbol{\epsilon} \in \mathbb{R}^{\mathscr{E}}$ there exists a velocity $\mathbb{w} \in \mathscr{U}$ such that*

$$\boldsymbol{\pi}_b \cdot \mathbb{w} = \boldsymbol{\epsilon}_b \text{ for every } b \in \mathscr{B}, \tag{4.47a}$$

$$\boldsymbol{\pi}_c \cdot \mathbb{w} \leq \boldsymbol{\epsilon}_c \text{ for every } c \in \mathscr{C}, \tag{4.47b}$$

if and only if for every proper prestress $\boldsymbol{\omega}$

$$\boldsymbol{\omega} \cdot \boldsymbol{\epsilon} \geq 0. \tag{4.48}$$

4.4 Sufficient Conditions Based on Expansions of the Motion

We follow the formulation of the second-order stress test, but using the full expansion of motions as in Subsection 3.3. As in that section, we will assume the placement we consider is in Rep. We saw there that a motion was admissible if it satisfies recursion relations of the form

$$2\boldsymbol{\pi}_e \cdot \mathbb{q}_s \leq -\sum_{r=1}^{s-1} B_e\, \mathbb{q}_r \cdot \mathbb{q}_{s-r} \qquad s = 1, 2, \ldots. \tag{4.49}$$

In these equations we stipulate that the $\leq$ converts to an equality for bars, and have not explicitly noted the possible truncation of the series for e a cable. To do the latter precisely, let us define a *sequence of un-shortened cables* recursively:

$$\mathscr{C}_0 = \mathscr{C} \tag{4.50a}$$

and for positive m

$$\mathscr{C}_m = \mathscr{C}_{m-1} \setminus \{\, c \in \mathscr{C}_{m-1} \mid 2\boldsymbol{\pi}_c \cdot \mathbb{q}_m < -\sum_{r=1}^{m-1} B_c\, \mathbb{q}_r \cdot \mathbb{q}_{m-r} \,\} \tag{4.50b}$$

We now can apply Proposition 4.13 to the set of inequalities.

Proposition 4.21. *Let $\mathbb{q}(t) := \sum t^s \mathbb{q}_s$ be a motion in Rep. Define the sequence $(\mathscr{C}_m)$ as in (4.50) and set $\mathscr{E}_m = \mathscr{B} \cup \mathscr{C}_m$. Then necessary and sufficient conditions that the expansion represent an admissible motion are that for each proper prestress $\boldsymbol{\omega}$,*

$$\forall e \in \mathscr{E} \qquad \boldsymbol{\pi}_e \cdot \mathbb{q}_1 \leq 0 \tag{4.51a}$$

and for $n = 1 \ldots \infty$

$$\sum_{e \in \mathscr{E}_{2n}} \left\{ \sum_{p=1}^{n-1} 2\, \omega_e\, B_e \mathbb{q}_p \cdot \mathbb{q}_{2n-p} + \omega_e\, B_e \mathbb{q}_n \cdot \mathbb{q}_n \right\} \leq 0 \tag{4.51b}$$

$$\sum_{e \in \mathscr{E}_{2n+1}} \left\{ \sum_{p=1}^{n} \omega_e\, B_e \mathbb{q}_p \cdot \mathbb{q}_{2n+1-p} \right\} \leq 0. \tag{4.51c}$$

In each of these equations, the $\leq$ *connotes equality for edges which are bars.*

An interesting consequence is applicable mostly to bar structures;

Corollary 4.22. *It there is no prestress a placement of a structure is stable if and only if it is rigid.*

It may be useful to examine the form of the first few terms from (4.51).

$$\begin{aligned} \sum \omega_e\, B_e\, \mathbb{q}_1 \cdot \mathbb{q}_1 &\leq 0 \\ \sum \omega_e\, B_e\, \mathbb{q}_1 \cdot \mathbb{q}_2 &\leq 0 \\ 2 \sum \omega_e\, B_e\, \mathbb{q}_1 \cdot \mathbb{q}_3 + \sum \omega_e\, B_e\, \mathbb{q}_2 \cdot \mathbb{q}_2 &\leq 0 \\ \sum \omega_e\, B_e\, \mathbb{q}_1 \cdot \mathbb{q}_4 + \sum \omega_e\, B_e\, \mathbb{q}_2 \cdot \mathbb{q}_3 &\leq 0 \end{aligned} \tag{4.52}$$

Proposition 4.21 carries a set of stopping rules or **sufficient conditions for stability**:

Corollary 4.23. *The placement is stable if for any expansion there is a prestress for which for some* $n > 1$

$$\sum_{\mathscr{E}_{2n}} \left\{ \sum_{p=1}^{n-1} 2\, \omega_e\, B_e \mathbb{q}_p \cdot \mathbb{q}_{2n-p} + \omega_e\, B_e \mathbb{q}_n \cdot \mathbb{q}_n \right\} > 0 \tag{4.53a}$$

or

$$\sum_{\mathscr{E}_{2n+1}} \left\{ \sum_{p=1}^{n} \omega_e\, B_e \mathbb{q}_p \cdot \mathbb{q}_{2n+1-p} \right\} > 0. \tag{4.53b}$$

These generalize the second-order stress test. The first few are:

$$\sum \omega_e B_e \mathbb{q}_1 \cdot \mathbb{q}_1 > 0 \tag{4.54a}$$
$$\sum \omega_e B_e \mathbb{q}_1 \cdot \mathbb{q}_2 > 0 \tag{4.54b}$$
$$2 \sum \omega_e B_e \mathbb{q}_1 \cdot \mathbb{q}_3 + \sum \omega_e B_e \mathbb{q}_2 \cdot \mathbb{q}_2 > 0 \tag{4.54c}$$
$$\sum \omega_e B_e \mathbb{q}_1 \cdot \mathbb{q}_4 + \sum \omega_e B_e \mathbb{q}_2 \cdot \mathbb{q}_3 > 0, \tag{4.54d}$$

or, in terms of the stress operator

$$\Omega \mathbb{q}_1 \cdot \mathbb{q}_1 > 0 \tag{4.55a}$$
$$\Omega \mathbb{q}_1 \cdot \mathbb{q}_2 > 0 \tag{4.55b}$$
$$2 \Omega \mathbb{q}_1 \cdot \mathbb{q}_3 + \Omega \mathbb{q}_2 \cdot \mathbb{q}_2 > 0 \tag{4.55c}$$
$$\Omega \mathbb{q}_1 \cdot \mathbb{q}_4 + \Omega \mathbb{q}_2 \cdot \mathbb{q}_3 > 0. \tag{4.55d}$$

Satisfaction of any one ensures that the expansion cannot be continued.

It is easy to deal with the case in which the motion might be of higher degree. For example, if $\mathbb{q}_i = 0$ for all $i < n$ then $\omega_e B_e \mathbb{q}_n \cdot \mathbb{q}_n > 0$ ensures stability.

5 Energies and Stability

The second-order stress test (Corollary 4.16) is a positivity condition which ensures stability; the results of the last section enlarge upon it. Here we shall expand upon this, developing ideas due to Connelly [10, 13] and to Calladine and Pellegrino from a different point of view.

5.1 Elastic Energies

We consider the equilibria of structures in which all of the edges are elastic. Placements in which such a structure is in equilibrium under null external loading will also serve as equilibrated placements for a non-elastic tensegrity structure of corresponding edge lengths (let us agree to call this the **hard** version of the structure). Physically, it seems clear that if motions from a placement of an elastic structure increase the energy of the system then both it and the corresponding hard structure will be stable. The converse clearly is false: a hinged rod under compression between fixed pivots is stable as a hard structure, but unstable as an elastic system.

Even in an elastic structure it is important to distinguish cables from bars. In this setting a cable is an elastic element which cannot accept stress when shorter than its natural length.

Recall that λ_e is half of the square of the length of the edge e. We let

$$l_e = \sqrt{2\lambda_e} \tag{5.1}$$

denote the **edge-length**. Then an elastic energy for the edge is specified by giving a **natural length** l_e^o and an **energy function** ψ_e. The energy of the edge is then given as

$$\psi_e(l_e - l_e^o) \tag{5.2}$$

with the provision that for cables c

$$\psi_c(l) = 0 \quad \text{for } l \le 0. \tag{5.3}$$

The simplest form of such a function is quadratic (if the edge is linearly elastic): $\psi_e(l) = \kappa_e\, l^2$. We shall assume, as is reasonable on physical grounds, that for all e

$$\psi_e(0) = 0, \tag{5.4}$$

$$\psi_e'(0) = 0, \tag{5.5}$$

$$\psi_b'' > 0 \quad \text{for bars and} \quad \psi_c''\Big|_{\mathbb{R}^-} > 0 \quad \text{for cables.} \tag{5.6}$$

It follows that each ψ_e is a convex function with non-negative values.

For our purposes it is more convenient to revert to using λ_e; from(5.1) the energy as a function of λ_e is

$$\phi_e(\lambda_e) = \psi_e(l_e - l_e^o) = \psi_e\left(\sqrt{2\lambda_e} - l_e^o\right) . \tag{5.7}$$

We relate this to our previous notations by equating the rate of change of energy to the rate of working in the edge. For a motion with initial velocity $\mathbb{v}$

$$\frac{d}{dt}\phi_e\Big|_{t=0} = \phi_e' \nabla\lambda_e \cdot \mathbb{v} = \phi_e' \boldsymbol{\pi}_e \cdot \mathbb{v} = \omega_e\, \epsilon_e, \tag{5.8}$$

which shows that in this context the stress in e is

$$\omega_e = \phi_e'(\lambda_e). \tag{5.9}$$

This is consistent with the interpretation of ω_e as a force per length, since

$$\phi_e'(\lambda_e) = \frac{\psi_e'(l_e - l_e^o)}{l_e}, \tag{5.10}$$

and $\psi_e'(l_e - l_e^o)$ is the force in the element at this value of length.

Lemma 5.1.

- *If* $l_e(\mathbb{p})$ *is greater than the natural length then* $\omega_e = \phi'(\lambda_e) > 0$
- *If* $l_e(\mathbb{p})$ *is less than the natural length then* $\omega_e = \phi'(\lambda_e) < 0$
- *If* $l_e(\mathbb{p})$ *is equal to the natural length then* $\omega_e = \phi'(\lambda_e) = 0$

The **total elastic energy** is

$$\Phi(\mathbb{p}) = \sum \phi_e(\lambda_e(\mathbb{p})). \tag{5.11}$$

We are concerned with the behavior of this function. If it is a strict local minimum at a placement, then the elastic structure is stable.

Proposition 5.2. *Suppose that* $\mathbb{p}$ *represents a strict local minimum of* Φ *modulo rigid motions. Then if all cables have non-negative stresses (*$\phi_c' \geq 0$*),* $\mathbb{p}$ *is a stable tensegrity placement for the corresponding tensegrity structure.*

Proof. To begin, note that since $\mathbb{p}$ gives a relative local minimum, the energy is stationary there:

$$\nabla\Phi = \sum \phi_e'(\lambda_e(\mathbb{p}))\nabla\lambda_e(\mathbb{p}) = \sum \omega_e \boldsymbol{\pi}_e = 0\,. \tag{5.12}$$

Thus the stresses in the elastic structure are (proper) prestresses for the hard structure.

Any placement $\mathbb{q}$ near $\mathbb{p}$ and admissible for the hard structure must have $\lambda(\mathbb{q}) \leq \lambda(\mathbb{p})$. Thus cables c have lengths at these $\mathbb{q}$ no longer than those at $\mathbb{p}$. Since the ψ_c are increasing functions, it follows that $\phi_c(\lambda_c(\mathbb{q})) \leq \phi_c(\lambda_c(\mathbb{p}))$. On the other hand, the lengths of bars b do not change, and so $\phi_b(\lambda_b(\mathbb{q})) = \phi_b(\lambda_b(\mathbb{p}))$. Hence $\Phi(\mathbb{q}) \leq \Phi(\mathbb{p})$. By hypothesis the inequality can only be an equality and $\mathbb{q}$ can differ from $\mathbb{p}$ only by a rigid transformation. $\square$

Thus a stable placement of an elastic tensegrity structure also is stable for the hard structure. But it is clear that less is required for the latter:

Corollary 5.3. *Suppose that* $\mathbb{p}$ *represents a strict local minimum of* Φ *among all motions which are admissible for the hard structure, modulo rigid motions. Then if all cables have non-negative stresses (*$\phi_c' \geq 0$*),* $\mathbb{p}$ *is a stable tensegrity placement for the hard tensegrity structure.*

Let us investigate conditions that ensure the energy is minimal. We have observed that (5.12) ensures that it is stationary.[3]

[3] If there are applied conservative forces, we add a potential for the applied forces to the energy; minimization then yields the full force balance (4.6).

Next, we examine the second-order terms. The second derivative is

$$\nabla^2\Phi = \sum \phi_e'' \boldsymbol{\pi}_e \otimes \boldsymbol{\pi}_e + \sum \omega_e B_e = \sum \phi_e'' \boldsymbol{\pi}_e \otimes \boldsymbol{\pi}_e + \Omega. \tag{5.13}$$

If we examine the values of this quadratic form on initial velocities, we note that it is zero on any rigid velocity. Thus an immediate criterion is

Corollary 5.4. *A sufficient condition for stability of a placement of an elastic tensegrity structure is that*

$$\sum \Phi_e'' [\boldsymbol{\pi}_e \cdot \mathbb{v}]^2 + \mathbb{v} \cdot \Omega \mathbb{v} > 0 \tag{5.14}$$

for all velocities $\mathbb{v} \in \mathcal{R}^\perp$.

Of course, we also have

Corollary 5.5. *A sufficient condition for stability of a placement of the hard version of an elastic tensegrity structure is that*

$$\sum \Phi_e'' [\boldsymbol{\pi}_e \cdot \mathbb{v}]^2 + \mathbb{v} \cdot \Omega \mathbb{v} > 0 \tag{5.15}$$

for non-rigid admissible velocities $\mathbb{v} \in \mathcal{R}^\perp$.

This **energy criterion** *is weaker, in general, than the second order stress test, but is identical if the only admissible velocities are isometric.* That condition is ensured if we know that the prestress leaves all cables stressed (Theorem 4.11).

Corollary 5.6. *If it is true that* $\phi_c'(\mathbb{p}) > 0$ *for all cables, then for any admissible velocity* $\mathbb{v}$

$$\nabla^2\Phi[\mathbb{v}, \mathbb{v}] = \mathbb{v} \cdot \Omega \mathbb{v} \tag{5.16}$$

so that the energy is positive-definite on the subspace of non-rigid admissible velocities if and only if Ω *is.*

To interpret (5.15) in physical terms, note that

$$\Phi_e'' = \frac{\psi_e''}{2\lambda_e} - \frac{\psi_e'}{(2\lambda_e)^{3/2}} \tag{5.17}$$

$$= \frac{\psi_e''}{l_e^2} - \frac{\psi_e'}{l_e^3} = \left(l_e \, \phi_e'' - \phi_e' \right) / l_e^3. \tag{5.18}$$

This is indeterminant in general, but in particular if the edge is linearly elastic, so that $\psi'_e = \kappa_e(l_e - l_e^o)$ and $\psi'_e = \kappa_e$, then

$$\Phi''_e = \kappa_e\, l_e^o / l_e^3 > 0. \tag{5.19}$$

and the term is positive, enhancing the second-order stability condition.
Remark We have chosen to "elastify" all of the edges of the structure. Similar calculations apply to one in which only some edges are taken to be elastic (*cf* [44]). In this case, the lengths of the hard edges enter the minimization as constraints, and their stresses as LaGrange multipliers. In this sense a hard structure is one in which the system consists only in constraints, with a null objective function.

5.2 Prestress Stability

Connelly and Whiteley take a constructive approach to the above energy calculations. They consider a tensegrity pair ($\mathbb{p}$, $\boldsymbol{\omega}$) and assert that $\mathbb{p}$ is stable if an energy function, or, more precisely, the quadratic localization of one, can be constructed. They attain more generality by allowing indefiniteness in the quadratic form.

In [13] they define a placement to be **prestress stable** if there is a proper prestress $\boldsymbol{\omega}$ and a family of non-negative numbers (γ_e) such that

- $H = \sum \gamma_e \boldsymbol{\pi}_e \otimes \boldsymbol{\pi}_e + \Omega$ is positive semi-definite,
- for any cable c, $\omega_c = 0 \Rightarrow \gamma_c = 0$,
- $H(\mathbb{v}, \mathbb{v}) = 0$ only if $\mathbb{v} \in \mathscr{R}$.

In other words they propose to construct an energy function which will serve the desired role in establishing stability. The restriction placed upon the stress-free cables serves to generalize applicability somewhat, as it enables ignoring of these cables (so long as their being non-stressed does not enable motions which leave H unchanged). They then observe that prestress stability implies stability.

6 Stability and Stresses

The results in the last sections dealt with local approximations. Some results which are purely topological were established by Connelly and Whiteley.

6.1 Existence of a Prestress

In [10] Connelly used relaxation techniques to relate stability and existence of a prestress for a tensegrity structure (we assume throughout this subsection that the structure has at least one cable). Our proof of these results is adapted from Whiteley [63]. The preliminary result presented next is of interest in itself. It indicates in particular that a strict prestress is in some sense generic for stable placements.

Proposition 6.1. *Let $\mathbb{p}$ be a stable placement of a tensegrity structure. Then within any neighborhood of $\mathbb{p}$ there is a placement with a strict proper prestress.*

Recall that strict means that there is a prestress on all cables; no assertion is made about bars.

Proof. Using Proposition 3.11, it suffices to work within Rep. Given $\mathbb{p}$, for each edge e we set $\Lambda_e = \boldsymbol{\lambda}_e(\mathbb{p})$ and construct a family of energy functions. For each bar b, set

$$f_b(\lambda) = (\lambda - \Lambda_b)^2. \tag{6.1}$$

For each cable c we construct a family of functions. Given $\delta \geq 0$, set

$$f_c^\delta(\lambda) = \begin{cases} \frac{\delta}{\Lambda_c}\lambda & 0 \leq \lambda < (1+\delta)\Lambda_c \\ (\delta + \delta^2) + \frac{\delta}{\Lambda_c}[\lambda - (1+\delta)\Lambda_c] & \\ +[\lambda - (1+\delta)\Lambda_c]^2 & \text{otherwise} \end{cases} \tag{6.2}$$

Thus f_c^0 is zero in $[0, \Lambda_c]$ and quadratic after, while each f_c^δ is linear, with positive slope, in $[0, \Lambda_c + \delta]$ and quadratic after. All are C^1, and the f_c^δ converge uniformly to f_c^0 on bounded intervals. We define the total energies as

$$H^\delta(\mathbb{q}) = \sum_{\mathscr{C}} f_c^\delta(\lambda_c(\mathbb{q})) + \sum_{\mathscr{B}} f_b(\lambda_b(\mathbb{q})) \tag{6.3}$$

Since $\mathbb{p}$ is stable, no placement $\mathbb{q}$ within a sufficiently small neighborhood simultaneously has $\lambda_b(\mathbb{q}) = \Lambda_b$ for all bars and $\lambda_c(\mathbb{q}) \leq \Lambda_c$ for all cables. Hence, for all such placements

$$H^0(\mathbb{q}) > H^0(\mathbb{p}) = 0. \tag{6.4}$$

Choose $\varepsilon > 0$ such that the ball about $\mathbb{p}$ of radius ε , $\mathbb{B}_\varepsilon(\mathbb{p})$, lies within the neighborhood which ensures (6.4) and set

$$m_\varepsilon = \min\{\, H^0(\mathbb{q}) \,|\, \mathbb{q} \in \partial\mathbb{B}_\varepsilon(\mathbb{p})\}. \tag{6.5}$$

Since each $\boldsymbol{\lambda}_e$ is quadratic,

$$\boldsymbol{\Lambda}(\mathbb{B}_\varepsilon(\mathbb{p})) - \boldsymbol{\Lambda}(\mathbb{p}) \tag{6.6}$$

is bounded. Hence H^δ converges uniformly to H^0 in the closed ball, and we may choose Δ so that

$$\delta < \Delta \;\Rightarrow\; |\, H^\delta(\mathbb{q}) - H^0(\mathbb{q})\,| < m_\varepsilon/2 \tag{6.7}$$

for each $\mathbb{q}$ in the closed ball. This means, in particular, that each H^δ has a minimal value on $\partial\mathbb{B}_\varepsilon(\mathbb{p})$ greater than $m_\varepsilon/2$ for all $\delta < \Delta$.

Finally, we argue that the minimum of some H^δ occurs at an interior point of $\mathbb{B}_\varepsilon(\mathbb{p})$. For any δ we can find a neighborhood of $\mathbb{p}$ within this ball such that each placement $\mathbb{q}$ in the neighborhood yields

$$\boldsymbol{\lambda}_c(\mathbb{q}) < (1+\delta)\,\Lambda_c \quad \text{and} \quad \boldsymbol{\lambda}_b(\mathbb{q}) < \Lambda_b + \delta \tag{6.8}$$

so that

$$H^\delta(\mathbb{q}) < \sum_{\mathscr{C}} (\delta + \delta^2) + \sum_{\mathscr{B}} \delta^2 = (\#\mathscr{C})\delta + (\#\mathscr{E})\delta^2. \tag{6.9}$$

We have supposed $\delta < \Delta$ and we may reduce it further to ensure that the bound in (6.9) is less than $m_\varepsilon/2$. Since points in the interior of $\mathbb{B}_\varepsilon(\mathbb{p})$ yield smaller values for H^δ than those on the boundary, the minimum occurs in the interior. The function is smooth, so at the minimizer

$$0 = \nabla H^\delta(\hat{\mathbb{q}}) = \sum_{\mathscr{C}} {f_c^\delta}'(\lambda_c(\hat{\mathbb{q}}))\,\boldsymbol{\pi}_c(\hat{\mathbb{q}}) + \sum_{\mathscr{B}} f_b{}'(\lambda_b(\hat{\mathbb{q}}))\,\boldsymbol{\pi}_b(\hat{\mathbb{q}}). \tag{6.10}$$

This list of derivatives provides a prestress for the structure at the point. By construction of the functions f_c^δ, the derivative of each always is positive, so that this prestress is proper and strict[4]. □

It is unlikely that the minimizing placement in the Proposition is $\mathbb{p}$. So we have not deduced that it must have a strict prestress. In fact one can easily find counterexamples, *ie* stable placements which do not have a strict prestress. However, it quickly follows that $\mathbb{p}$ has a non-zero prestress.

Theorem 6.2. *If $\mathbb{p}$ is a stable placement for a tensegrity structure, then there is a non-zero proper prestress for $\mathbb{p}$.*

[4]Note that if $\mathscr{C}$ were empty, we could not conclude that some stresses are non-zero.

Proof. For each sufficiently large integer n we use the previous result to find a $\mathbb{q}_n$ with $\|\mathbb{q}_n - \mathbb{p}\| < 1/n$ which has a strict prestress. Since prestresses are homogeneous, we can choose each prestress to be on the unit sphere in $\mathbb{R}^{\mathscr{E}}$. The sequence of prestresses $\{\boldsymbol{\omega}_n\}$ on this compact set has a convergent subsequence. Choosing this subsequence, without change in notation, we arrive at a convergent sequence

$$(\mathbb{q}_n, \boldsymbol{\omega}_n) \longrightarrow (\mathbb{p}, \boldsymbol{\omega}_\infty)). \tag{6.11}$$

The map

$$(\mathbb{q}, \omega) \mapsto \Pi(\mathbb{q})\boldsymbol{\omega} \tag{6.12}$$

is continuous, so that $\Pi(\mathbb{p})\boldsymbol{\omega}_\infty = \mathbf{0}$, *ie*, $\boldsymbol{\omega}_\infty$ is a prestress for $\mathbb{p}$. It is on the unit sphere, and thus non-zero, and since $(\boldsymbol{\omega}_n)_c > 0$ for all c, the stress which $\boldsymbol{\omega}_\infty$ assigns to each cable is non-negative. □

6.2 The Stress Operator and Stability

We introduced the stress operator as the symmetric linear operator on $\mathbb{R}^{3\mathcal{N}}$ given by

$$\Omega = \sum \omega_e B_e \tag{6.13}$$

We formalize the associated quadratic form as

$$\sigma_\omega(\mathbb{q}) = \frac{1}{2}\,\mathbb{q} \cdot \Omega\,\mathbb{q}\,. \tag{6.14}$$

Here we have used a subscript to emphasize that the potential depends upon a specified stress vector. Later, we will suppress this notation, if ω is understood.

Introduction of Ω allows a dual approach to stability. One may consider ω as the fixed element and ask which $\mathbb{p}$ one can associate to it.

The stress potential figures in the second-order stress test (Corollary 4.15); we would like to exploit the corresponding sufficient condition (Corollary 4.17). We acquire some tools for its utilization in the following. (Recall that a pair $(\mathbb{p}, \omega)$ is called a tensegrity pair if $\boldsymbol{\omega}$ is a prestress for $\mathbb{p}$.)

Proposition 6.3. *Given a prestress ω and the associated stress potential σ*

1. *For any placement* $\mathbb{q}$

$$\sigma(\mathbb{q}) = \frac{1}{2}\sum \boldsymbol{\omega}_e B_e \mathbb{q} \cdot \mathbb{q} = \sum \boldsymbol{\omega}_e B_e \mathbb{q} \cdot B_e \mathbb{q} = 2\sum \boldsymbol{\omega}_e \boldsymbol{\lambda}_e(\mathbb{q}) = 2\boldsymbol{\omega} \cdot \boldsymbol{\lambda}(\mathbb{q}).$$

2. *σ is* **Euclidean invariant**: *for any orthogonal Q, any vectors* $\mathbf{x}$ *and* $\mathbf{y}$

$$\sigma(Q(\mathbb{q}-\mathbf{x})+\mathbf{y}) = \sigma(\mathbb{q})$$

for all $\mathbb{q}$.

3. $(\mathbb{p},\omega)$ *is a tensegrity pair iff*

$$\nabla\sigma(\mathbb{p}) = \Omega\mathbb{p} = \mathbf{0}.$$

4. *If* $(\mathbb{p},\omega)$ *is a tensegrity pair then*

$$\sigma(\mathbb{p}) = 0,$$

which is the same as $\boldsymbol{\omega}\cdot\boldsymbol{\lambda}(\mathbb{p}) = 0$

5. *If* $(\mathbb{p},\omega)$ *is a tensegrity pair then for any placement* $\mathbb{q}$

$$\sigma(\mathbb{q}) = \sigma(\mathbb{q}-\mathbb{p}).$$

6. *If ω is a null stress for some placement, then the null space of Ω is at least 12-dimensional; hence σ cannot be positive-definite.*

The proofs are immediate. For the last item, we recall that the subspace of affine motions away from the tensegrity position has dimension 12.

Thus one quick conclusion is that each stress potential associated to a placement will be stationary, with value 0 at that placement. Were this a local minimum we could apply Corollary 4.17 to deduce that the placement is stable. In fact, this cannot occur for unconstrained structures. However Connelly [10] observes that this is true for a class of constrained structures which he aptly calls spider webs.

Proposition 6.4. *Let a placement of a (necessarily constrained) structure admit a strict positive prestress. Then it is stable.*

Proof. Consider the tensegrity pair $(\mathbb{p},\omega)$. For any placement $\mathbb{q}$ the value of the associated stress potential is

$$\sigma(\mathbb{q}) = \sum \boldsymbol{\omega}_e\boldsymbol{\lambda}_e(\mathbb{q}). \tag{6.15}$$

This means that if ω is positive and $\mathbb{q}$ is admissible then $\sigma(\mathbb{q}) \le \sigma(\mathbb{p})$. On the other hand,

$$\sigma(\mathbb{q}) = \sum \boldsymbol{\omega}_e\boldsymbol{\lambda}_e(\mathbb{q}-\mathbb{p}), \tag{6.16}$$

and if the two placements differ at all then $\sigma(\mathbb{q})$ is positive. Thus there can be no other admissible placements. □

Note that we have proved more than stated: $\mathbb{p}$ is unique among admissible placements.

Of course the restriction to positive prestress limits the direct applicability of this result; one expects such a result only for constrained systems, such as spider webs. Connelly and Whitely go on to exploit this in other ways.

6.3 The Reduced Stress Operator

We can utilize the reduced kinematics of subsection 2.4 to find a reduced form of the stress operator. From the representation of B_e in (2.37) we have

$$\Omega = \sum_e \omega_e B_e \leftrightarrow \left(\sum_e \omega_e C_e \right) \otimes \mathbb{1} = \widehat{\Omega} \otimes \mathbb{1}, \tag{6.17}$$

where $\widehat{\Omega}$ is the **reduced stress operator** ([13]), an automorphism of $\mathbb{R}^{\mathcal{N}}$. Its $\#\mathcal{N} \times \#\mathcal{N}$ matrix can easily be seen to be sparse, with diagonal entries the sum of the stresses at the appropriate node, and off-diagonal entries $-\omega_e$ at the locations involving the end-nodes of e.

Since we replace $B_e \mathbb{q}$ by

$$(C_e \otimes \mathbb{1}) \sum_{\alpha \in \mathcal{N}} \rho_\alpha \otimes \mathbb{q}_\alpha = \sum_{\alpha \in \mathcal{N}} C_e \rho_\alpha \otimes \mathbb{q}_\alpha \tag{6.18}$$

we have

$$(\sum_{\beta \in \mathcal{N}} \rho_\beta \otimes \mathbb{q}_\beta) \cdot ((C_e \otimes \mathbb{1}) \sum_{\alpha \in \mathcal{N}} \rho_\alpha \otimes \mathbb{q}_\alpha) = \sum_{\beta \in \mathcal{N}} \sum_{\alpha \in \mathcal{N}} (C_e \rho_\alpha \cdot \rho_\beta)(\mathbb{q}_\alpha \cdot \mathbb{q}_\beta) \tag{6.19}$$

and thus

$$\sigma_\omega(\mathbb{q}) = \sum_{\beta \in \mathcal{N}} \sum_{\alpha \in \mathcal{N}} (\widehat{\Omega} \rho_\alpha \cdot \rho_\beta)(\mathbb{q}_\alpha \cdot \mathbb{q}_\beta) \tag{6.20}$$

We find that *the symmetric quadratic form σ is determined by the lower-dimensional symmetrical quadratic form generated by $\widehat{\Omega}$*. This is an advantage in computations.

7 Kinematic Constraints

In engineering applications it often occurs that one or more nodes of the structure is constrained to move in particular ways. For example, one or

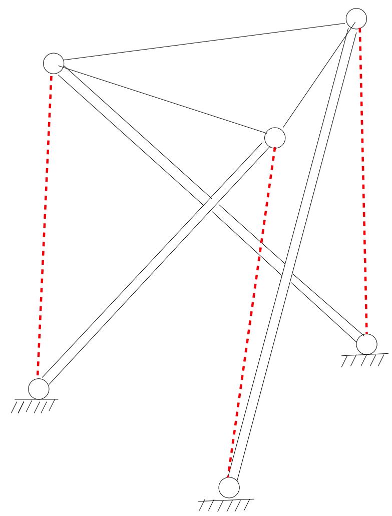

Figure 4: Constrained T-3 Structure

more may be constrained to a fixed position or to move only along a line or within a plane; in Figure 4 we consider a constrained version of T3. A constraint of this sort may serve as a means to eliminate the possibility of rigid motion of the structure, but, as is the case pictured here, often the constraint is more restrictive. We consider kinematic constraints of the form

$$\mathbb{p} \in \mathbb{b} + \mathcal{X} \quad \mathbb{b} \in \mathcal{X}^{\perp} \tag{7.1}$$

restricting placements to an affine subspace of $\mathbb{R}^{3\mathcal{N}}$. It follows that the admissible velocity space now should be restricted to $\mathcal{X}$. For a constrained structure the admissible class of rigid body motions consist in those that which respect (7.1), and hence we define the space of **restricted rigid velocity fields** as

$$\mathcal{R}_{\mathcal{X}} := \mathcal{X} \cap \mathcal{R}(\mathbb{p}). \tag{7.2}$$

Its orthogonal complement in $\mathcal{X}$ is

$$\mathcal{R}_{\mathcal{X}}{}^{\perp} := \mathcal{X} \cap (\mathcal{R}_{\mathcal{X}})^{\perp} = \mathcal{X} \cap \left(\mathcal{R}^{\perp} + \mathcal{X}^{\perp} \right). \tag{7.3}$$

We define the orthogonal projection

$$P : \mathbb{R}^{3\mathcal{N}} \to \mathcal{X}, \tag{7.4}$$

and note that

Lemma 7.1. *For any subspace* $\mathscr{W}$*,*

$$P\mathscr{W} = \mathfrak{X} \cap \left(\mathfrak{X}^{\perp} + \mathscr{W} \right). \tag{7.5}$$

It follows that

$$\mathscr{R}_{\mathfrak{X}}{}^{\perp} = P\left(\mathscr{R}^{\perp} \right). \tag{7.6}$$

Even if the placement is restricted by (7.1), the edge vectors $\boldsymbol{\pi}_e$ need not be in $\mathfrak{X}$, so we define the **constrained edge vectors** and the **constrained geometric matrix** as

$$\boldsymbol{\psi}_e := P\boldsymbol{\pi}_e; \quad \Psi := P\Pi. \tag{7.7}$$

Notice that if a node is restricted not to move then the corresponding row in Π is eliminated by the action of P; if the constraint renders an edge immobile then the corresponding column vector still is present, but in practice might as well be omitted. We see then that

Corollary 7.2. *Each constrained edge vector is in* $\mathscr{R}_{\mathfrak{X}}{}^{\perp}$.

Also, it now follows that the constrained geometric matrix is the transpose of the gradient of the length function λ as restricted to the subspace (7.1).

Working within the constrained subspaces, the notions of stability and rigidity can be be redefined and it is immediate that

Proposition 7.3. *If a structure described by* Π *is rigid then the constrained structure described by* $P\Pi$ *is rigid; if it is stable then so is the constrained structure.*

Naturally, the converse is false.

Next, we turn to the forces. A **reaction force** $\mathbb{r}$ is an external force which does no work under any admissible velocity field, that is, $\mathbb{r} \in \mathfrak{X}^{\perp}$. For example, if a node is fixed in space, every force applied to that node is a reaction force, while if it is constrained to travel in one direction, reaction forces are external forces normal to that one direction. Reaction forces are generated by the constraint exactly sufficiently to maintain the constraint. For constrained structures we distinguish **applied forces** which lie in $\mathfrak{X}$ from the total external force, which includes reaction forces.

An applied force is said to be $\mathfrak{X}$**-equilibrated** if it is in $\mathscr{R}_{\mathfrak{X}}{}^{\perp}$ and we have that such a force always is equilibrated in the global sense by reaction forces:

Proposition 7.4. *If the applied force* $\mathbb{f} \in \mathfrak{X}$ *is* $\mathfrak{X}$*-equilibrated, then there is a reaction force* $\mathbb{r} \in \mathfrak{X}^\perp$ *such that* $\mathbb{f} + \mathbb{r}$ *is equilibrated, ie, such that*

$$\mathbb{f} + \mathbb{r} \in \mathscr{R}^\perp. \tag{7.8}$$

This follows immediately from Lemma 7.1 Of course, any equilibrated external force resolves into an equilibrated applied force plus a reaction force. Since the constrained edge vectors are in $\mathscr{R}_{\mathfrak{X}}{}^\perp$, it follows that any stressing of the structure results in an equilibrated applied force, and the definition of statical rigidity can be modified to the constrained case.

By replacing $\mathbb{R}^{3\mathscr{N}}$ by $\mathfrak{X}$, one can easily see that the various results shown for the unconstrained system extend without change to the constrained one. In particular, we note that if $\mathbb{p} = \mathbb{b} + \mathbb{x}$ then

$$\boldsymbol{\psi}_e = B_e\mathbb{b} + B_e\mathbb{x}, \tag{7.9}$$

so that if we restrict to the constraint space the stress operator becomes

$$\widetilde{\Omega} = P\Omega P = \sum PB_eP\,, \tag{7.10}$$

and we operate with this on the subspace $\mathfrak{X}$.

8 Tensegrity and Rank-deficiency

8.1 Snelson Structures and Maxwell's Rule

Consider that

$$\mathrm{Rank}(\Pi) = \dim \mathrm{Span}\,\{\,\boldsymbol{\pi}_e \mid e \in \mathscr{E}\}\,; \tag{8.1a}$$

$$\mathrm{Span}\,\{\,\boldsymbol{\pi}_e \mid e \in \mathscr{E}\} \subset \mathscr{R}^\perp. \tag{8.1b}$$

Thus the rank of Π is at most the minimum of $\#\mathscr{E}$ and $3\#\mathscr{N} - 6$. In this section we consider the case in which the structure satisfies

$$\#\mathscr{E} \leq \dim \mathscr{R}^\perp = 3\#\mathscr{N} - 6\,, \tag{8.2}$$

ensuring that the list of edge vectors could be linearly independent. We say that such a structure is a **Snelson structure**, since this condition is satisfied by all of the classical constructions by him and many of those of Fuller. Their virtue, aesthetic to the one and efficient to the other, was that these were *minimal structures.*

Consider the case in which the set of edge vectors for the given placement is linearly independent. This ensures that there can be no prestress, and hence, by Theorem 6.2, that the placement cannot be stable.

Proposition 8.1. *If a Snelson structure is not a bar-truss, any placement in which the geometric matrix is of maximal rank is unstable.*

Given a placement $\mathbb{p}$, the set $\mathbb{V}_o$ of *isometric* admissible velocities decomposes into the sum of two subspaces. The first is the subspace of rigid velocities $\mathscr{R}$ and the second is $\mathbb{V}_o \cap \mathscr{R}^\perp$, which we may call the space of **flexures**. The latter has dimension $\dim \mathscr{R}^\perp - \mathrm{Rank}(\Pi)$, often called the *degree of flexure*. Similarly, the *degree of deficiency* is $\#\mathscr{E} - \mathrm{Rank}(\Pi)$, *ie*, is the dimension of the space of prestresses. The statement that the rank of the matrix Π is the same as that of its transpose is often called **Maxwell's rule** [5]:

$$\#\mathscr{E} - \#\text{modes of prestress} = 3\#\mathscr{N} - 6 - \#\text{modes of flexure} \tag{8.3}$$

or

$$\#\text{modes of flexure} - \#\text{modes of prestress} = 3\#\mathscr{N} - 6 - \#\mathscr{E}.$$

Since we assume that $\#\mathscr{E} \leq 3\#\mathscr{N} - 6$, we conclude that

Proposition 8.2. *If a Snelson tensegrity structure is stable, it is flexible, ie, admits a non-rigid velocity which preserves lengths.*

This is the characterizing property of Snelson tensegrity structures.

8.2 The Geometry of Rank-deficient Structures

We deal here with a Snelson structure.

We have had occasion to deal with the **length-invariance manifold**: given $\mathbb{p}$, it is

$$\mathscr{A}_{\mathbb{p}} = \{\, \mathbb{q} \in \mathbb{R}^{3\mathscr{N}} \mid \lambda(\mathbb{q}) = \lambda(\mathbb{p}) \,\} . \tag{8.4}$$

This closed subset of $\mathbb{R}^{3\mathscr{N}}$ is (locally) a differentiable manifold when the rank of $\Pi^\top = \nabla\boldsymbol{\lambda}$ is constant on $\mathscr{A}$ in a neighborhood of $\mathbb{p}$. If $\mathbb{p}$ is stable, then it must be true that some neighborhood of $\mathbb{p}$, $\mathscr{A}_{\mathbb{p}}$ includes only elements of $\mathrm{Euc}(\mathbb{p})$ and hence that this open set is a differentiable manifold whose tangent space at $\mathbb{p}$ consists exactly in $\mathscr{R}(\mathbb{p})$.

A different concept is that of the rank-deficiency manifolds. First, we define

$$\mathfrak{P} = \left\{ \mathbb{q} \in \mathbb{R}^{3\mathscr{N}} \mid \mathrm{Rank}(\Pi(\mathbb{q})) = \dim \mathrm{Span}\big(\boldsymbol{\pi}_e(\mathbb{q})\big)_{e\in\mathscr{E}} < \#\mathscr{E} \right\} . \tag{8.5}$$

As we have observed, this closed set includes all stable placements. But also

Lemma 8.3. *Any neighborhood of a stable placement contains other points of* $\mathfrak{P}$.

This follows from Proposition 6.1: every neighborhood of the placement includes stressed placements.

The set $\mathfrak{P}$ partitions into subsets with specified rank, the **rank-deficiency manifolds**:

$$\mathfrak{P}_r = \{\, \mathbb{p} \in \mathfrak{P} \mid \operatorname{rank}(\Pi(\mathbb{p})) = \#\mathscr{E} - r\}, \quad 6 \leq r \leq \#\mathscr{E}\,. \tag{8.6}$$

We deduce the structure of the sets from that of the corresponding sets of singular matrices. We introduce a more generic notation to describe the latter. Let k and n be integers, with $k \leq n$. Then for each $s \leq k$ we introduce

$$\mathscr{M}_s = \{\, D \in \mathbb{R}^{n \times k} \mid \operatorname{rank}(D) = s\}. \tag{8.7}$$

We continue to write the matrix D in terms of its column vectors, staying with our notation

$$D = [\boldsymbol{\pi}_1 \cdots \boldsymbol{\pi}_k]\,. \tag{8.8}$$

The set $\mathscr{M}_k$ is an open set in the set of all $n \times k$ matrices (the **Steifel manifold**), but each of the smaller ones is a differentiable manifold of reduced dimension (**generalized Stiefel manifolds**). These were introduced by Milnor [36, 40], but since they do not seem to be well known, we will derive the formulae which we need. The simplest case, when the rank is $k-1$, is a model for the other calculations:

Lemma 8.4. $\mathscr{M}_{k-1}$ *is a differentiable manifold of dimension* $(k-1)(n+1)$. *Its tangent space at* D *consists of all* $n \times k$ *matrices orthogonal to*

$$\mathbb{v} \otimes \boldsymbol{\omega} \tag{8.9}$$

where $\boldsymbol{\omega}$ *is a non-zero vector in the null space of* D *and* $\mathbb{v}$ *ranges over all vectors in the null space of* $D^\top$.

Proof. To describe the tangent space: a matrix $D = [\boldsymbol{\pi}_1 \ldots \boldsymbol{\pi}_k]$ is in the manifold if the span of its column vectors is of dimension $k-1$ but the exterior product

$$\boldsymbol{\pi}_1 \wedge \ldots \wedge \boldsymbol{\pi}_k = 0. \tag{8.10}$$

Consider a path on the manifold passing through D. Taking its derivative at the base point delivers

$$\sum_i \boldsymbol{\pi}_1 \wedge \ldots \wedge \alpha_{\mathbf{i}} \wedge \ldots \wedge \boldsymbol{\pi}_k = 0, \tag{8.11}$$

where $\alpha_{\mathbf{i}}$, the derivative of $\boldsymbol{\pi}_i$, appears in the ith place in the list. One of the vectors $\boldsymbol{\pi}_l$ can be expressed as a linear combination of the others; to save notation, let us suppose it is the kth:

$$\boldsymbol{\pi}_k = \sum_{j}^{k-1} \mu_j \boldsymbol{\pi}_j, \tag{8.12}$$

so that ω has entries $[-\mu_1, \ldots, -\mu_n, 1]$. Then

$$\sum_{i}^{k} \sum_{j}^{k-1} \mu_j \boldsymbol{\pi}_1 \wedge \ldots \wedge \alpha_{\mathbf{i}} \wedge \ldots \wedge \boldsymbol{\pi}_j = 0. \tag{8.13}$$

Note that each exterior product is zero, due to repeated entries, except when $i = j$ or $i = k$. Thus we have

$$\boldsymbol{\pi}_1 \wedge \ldots \wedge \boldsymbol{\pi}_{k-1} \wedge \alpha_{\mathbf{k}} + \sum_{i}^{k-1} \mu_i \boldsymbol{\pi}_1 \wedge \ldots \wedge \alpha_{\mathbf{i}} \wedge \ldots \wedge \boldsymbol{\pi}_i = 0 \tag{8.14}$$

or

$$(\boldsymbol{\pi}_1 \wedge \ldots \wedge \boldsymbol{\pi}_{k-1}) \wedge \alpha_{\mathbf{k}} + \sum_{i}^{k-1} -(\boldsymbol{\pi}_1 \wedge \ldots \wedge \ldots \wedge \boldsymbol{\pi}_{k-1}) \wedge (\mu_i \alpha_{\mathbf{i}}) = 0. \tag{8.15}$$

But this says that

$$(\boldsymbol{\pi}_1 \wedge \ldots \wedge \boldsymbol{\pi}_{k-1}) \wedge \left(\alpha_{\mathbf{k}} - \sum_{i}^{k-1} \mu_i \alpha_{\mathbf{i}} \right) = 0, \tag{8.16a}$$

or, recalling the relation of the μ_i and ω,

$$(\boldsymbol{\pi}_1 \wedge \ldots \wedge \boldsymbol{\pi}_{k-1}) \wedge (A\omega) = 0, \tag{8.16b}$$

where $A = [\alpha_1 \ldots \alpha_k]$ is the derivative at the base point. (8.16b) means that $A\omega$ is in the span of the other vectors, *ie*, in the range of D, which can be expressed as saying that

$$\mathbb{v} \cdot A\omega = A \cdot (\mathbb{v} \otimes \omega) = 0 \tag{8.17}$$

for all vectors $\mathbb{v}$ in the null space of $D^\top$. $\square$

The proof extends, with only an increase in combinatorial complexity, to each of the manifolds $\mathscr{M}_s$ by considering the smaller submatrices and considering the exterior products of s-lists of column vectors. We obtain

Lemma 8.5. *$\mathscr{M}_s$ is a differentiable manifold of dimension $s(k+n-s)$. Its tangent space at D consists of all A orthogonal to*

$$\mathbb{v} \otimes \boldsymbol{\omega} \tag{8.18}$$

where $\boldsymbol{\omega}$ ranges over all vectors in the the null space of D and $\mathbb{v}$ ranges over all vectors in the null space of $D^\top$.

Now we pull this structure back to $\mathbb{R}^{3\mathscr{N}}$. Let us introduce the mapping

$$\mathbb{B} : \mathbb{R}^{3\mathscr{N}} \to \mathbb{R}^{3\mathscr{N}\times\mathscr{E}} \tag{8.19}$$

which assigns to each placement the corresponding Π; identifying $\mathbb{R}^{3\mathscr{N}\times\mathscr{E}}$ with $\mathbb{R}^{3\#\mathscr{N}\times\#\mathscr{E}}$, we see that

$$\mathfrak{P}_r = \mathbb{B}^{-1}(\mathscr{M}_{\#\mathscr{E}-r})\,. \tag{8.20}$$

Lemma 8.6. *The null space of $\mathbb{B}$ is*

$$N_{\mathbb{B}} := \{\, \mathbf{u} \in \mathbb{R}^{3\mathscr{N}} \mid \mathbf{u} \in \mathbb{R}^3 \}\,, \tag{8.21}$$

and hence is 3-dimensional. The range of $\mathbb{B}$, of dimension $3\#\mathscr{N}-3$, is

$$R_{\mathbb{B}} := \left\{ \sum_{\mathscr{E}} B_e \mathbb{w} \otimes \mathbf{e}_e \,\middle|\, \mathbb{w} \in \mathbb{R}^{3\mathscr{N}} \right\} \tag{8.22a}$$

$$= \left\{ \mathbb{u} \otimes \boldsymbol{\mu} \,\middle|\, \sum \boldsymbol{\mu}_e B_e \mathbb{u} = 0 \right\}^{\perp}\,. \tag{8.22b}$$

Here $(\mathbf{e}_e)_{e\in\mathscr{E}}$ is the standard basis of $\mathbb{R}^{\mathscr{E}}$.

Proof. $\mathbb{B}\mathbb{p} = \mathbf{0}$ says exactly that for each e, $B_e\mathbb{p} = \mathbf{0}$. But $B_e\mathbb{p} = \mathbf{0}$ if and only if $\mathbb{p}_{e_\alpha} = \mathbb{p}_{e_\omega}$. We have assumed our structures are connected, and hence each entry in $\mathbb{p}$ has to have the same value. Hence the null space has the indicated form.

Regarding the range, note that by definition

$$\mathbb{B}\mathbb{w} = \sum B_e \mathbb{w} \otimes \mathbf{e}_e$$

so (8.22a) is immediate. For the second form we need to represent $\mathbb{B}^\top$. Given $\mathbb{q} \in \mathbb{R}^{3\mathscr{N}}$ and $\mathbb{u} \otimes \boldsymbol{\mu} \in \mathbb{R}^{3\mathscr{N}\times\mathscr{E}}$, we find

$$\begin{aligned}
\mathbb{B}^\top(\mathbb{u}\otimes\boldsymbol{\mu}) \boldsymbol{\cdot} \mathbb{q} &= (\mathbb{u}\otimes\boldsymbol{\mu}) \boldsymbol{\cdot} \mathbb{B}\mathbb{q} \\
&= (\mathbb{u}\otimes\boldsymbol{\mu}) \boldsymbol{\cdot} \sum B_e \mathbb{q} \otimes \mathbf{e}_e \\
&= \sum \boldsymbol{\mu}_e B_e \mathbb{q} \boldsymbol{\cdot} \mathbb{u} = \sum \boldsymbol{\mu}_e B_e \mathbb{u} \boldsymbol{\cdot} \mathbb{q} \\
&= \left(\sum \boldsymbol{\mu}_e B_e \mathbb{u} \right) \boldsymbol{\cdot} \mathbb{q}\,.
\end{aligned}$$

Thus we can characterize the null space of $\mathbb{B}^\top$ as the span of those dyads $\mathbb{u} \otimes \boldsymbol{\mu}$ with $\left(\sum \boldsymbol{\mu}_e B_e \mathbb{u}\right) = \mathbf{0}$. The range of $\mathbb{B}$ is the orthogonal complement of that subspace. □

By (8.20) each $\mathfrak{P}_r$, if not empty, is a differentiable manifold, the preimage under $\mathbb{B}$ of the intersection of the manifold $\mathcal{M}_{\#\mathcal{E}-r}$ with the subspace $R_{\mathbb{B}}$. If this intersection is not empty (equivalent to $\mathfrak{P}_r$ being empty) it cannot consist of a single point. For if $\mathbb{B}\mathbb{p}$ is in the intersection, we recall that for any $L \in \mathrm{GL}(\mathbb{R}^3)$ and each $e \in \mathcal{E}$, $B_e L\mathbb{p} = LB_e\mathbb{p}$ so that $L\mathbb{B}\mathbb{p}$ is in $R_{\mathbb{B}}$. But as a linear endomorphism of $\mathbb{R}^{3\mathcal{N}}$, L still is invertible, so that $\mathbb{B}\mathbb{p} \in \mathcal{M}_{\#\mathcal{E}-r}$ implies $L\mathbb{B}\mathbb{p} \in \mathcal{M}_{\#\mathcal{E}-r}$, yielding a family of other points in the intersection.

Theorem 8.7. *If not empty, $\mathfrak{P}_r$ is a differentiable manifold in $\mathbb{R}^{3\mathcal{N}}$, whose tangent space at $\mathbb{p}$ is the set of vectors normal to the span of*

$$\sum_e \omega_e B_e \mathbb{v} = \Omega(\omega)\mathbb{v}, \tag{8.23}$$

where ω ranges over all prestresses and $\mathbb{v}$ ranges over all flexes.

Proof. First, note that elements of $N_{\mathbb{B}}$ satisfy the criterion (8.23) since each B_e annihilates all constant-entry vectors in $\mathbb{R}^{3\mathcal{N}}$.

The tangent space of $R_{\mathbb{B}} \cap \mathcal{M}_{\#\mathcal{E}-r}$ consists in those vectors in $R_{\mathbb{B}}$ normal to the various $\mathbb{v} \otimes \omega$ at the point. Hence the pull-back tangent space at a placement is

$$\begin{aligned}
\mathcal{T}_{\mathbb{p}}\mathfrak{P}_s &= \{\, \mathbb{w} \,|\, (\mathbb{v} \otimes \omega) \cdot \sum_e B_e \mathbb{w} \otimes \mathbf{e}_e = 0 \text{ for all } \omega,\ \mathbb{v}\} \\
&= \{\, \mathbb{w} \,|\, \sum_e \mathbb{v} \cdot B_e \mathbb{w}\, \omega_e = 0 \text{ for all } \omega,\ \mathbb{v}\} \\
&= \{\, \mathbb{w} \,|\, (\sum_e \omega_e B_e \mathbb{v}) \cdot \mathbb{w} = 0 \text{ for all } \omega,\ \mathbb{v}\}.
\end{aligned} \tag{8.24}$$

Since the sum $\sum_e \omega_e B_e \mathbb{v}$ is zero for all rigid velocities, the only *non-zero* combinations $\sum_e \omega_e B_e \mathbb{v}$ occur when $\mathbb{v}$ is a flex. □

Lemma 8.8. *The tangent space to $\mathfrak{P}_s$ at $\mathbb{p}$ includes the set of rigid velocities $\mathcal{R}(\mathbb{p})$.*

Proof. Given ω and $\mathbb{v}$ and $Q\mathbb{p} + \mathbf{r}$, we note

$$\begin{aligned}
\sum_e \omega_e B_e \mathbb{v} \cdot (Q\mathbb{p} + \mathbf{r}) &= \sum_e (\,\omega_e B_e \mathbb{v} \cdot Q\mathbb{p}\,) + \sum_e (\,\omega_e B_e \mathbb{v} \cdot \mathbf{r}) \\
&= Q\left(\sum_e \omega_e B_e \mathbb{p}\right) \cdot \mathbb{v} + \sum_e (\,\omega_e \mathbb{v} \cdot B_e \mathbf{r}) = 0 + 0
\end{aligned}$$

$\square$

We expand on the discussion just prior to the theorem to observe that the various manifolds $\mathfrak{P}_r$ are invariant under a simple group of automorphisms (*cf* [64]). Given $L \in \mathrm{GL}(\mathbb{R}^3)$ and $\mathbf{r} \in \mathbb{R}^3$, the affine automorphism $\mathbb{u} \mapsto L\mathbb{u} + \mathbf{r}$ factors as $B_e\,(L\mathbb{p} + \mathbf{r}^{\dagger}) = L\,B_e\,\mathbb{p}$. This then leads by easy computations to the following.

Proposition 8.9. $\mathfrak{P}_r$ *is invariant under the group of affine automorphisms described above, and*

- *the set of null stresses at* $L\mathbb{p} + \mathbf{r} \in \mathfrak{P}_r$ *is the same as that at* $\mathbb{p} \in \mathfrak{P}_r$*,*
- *the cone of admissible vectors and the space of isometric admissible vectors at* $L\mathbb{p} + \mathbf{r}$ *are given by* $L^{-\top}$ *acting upon the corresponding sets of vectors at* $\mathbb{p}$*, and*
- *the tangent space to* $\mathfrak{P}_r$ *at* $L\mathbb{p} + \mathbf{r}$ *is given by* L *acting on the tangent space at* $\mathbb{p}$*.*

The proof is straightforward.

Note that the invariance of the stress vector ω under affine automorphisms does not imply that the actual tensions and compressions in the members stay the same. The force carried by edge e is $\sqrt{\lambda_e}\omega_e$ and thus will change if the automorphism changes the length of the edges. The exception to this observation is the set of Euclidean isomorphisms; in particular we note that the rank-deficiency manifold passing through a placement $\mathbb{p}$ is an isomorphic image of that passing through the corresponding fixed placement in Rep, and the two have the same member-tensions and -compressions.

8.3 A Marching Algorithm

The characterization of the tangent space to $\mathfrak{P}_r$ can prove useful in describing traversal of the surface through manipulations of edge-lengths of a structure. Let us consider a special case, one which describes all of the structures built by Snelson. We assume that the structure satisfies

$$\#\mathscr{E} = 3\#\mathscr{N} - 6, \tag{8.25}$$

so that it is truely optimal. To avoid the inconvenience of dealing with the representation of the six coordinates describing the rigid motions, let us work in Rep.

Consider a placement $\mathbb{p} \in \mathfrak{P}_1$ (so that there is "only one" prestress $\boldsymbol{\omega}$ and flex $\mathbb{v}$.

Proposition 8.10. *If at $\mathbb{p} \in \mathfrak{P}_1$ for each admissible flex we have*

$$\mathbb{v} \cdot \sum \omega_e B_e \mathbb{v} \neq 0 \tag{8.26}$$

then there is a coordinate system in a neighborhood of $\mathbb{p}$ consisting of edge-lengths.

Proof. We denote the normal to $\mathfrak{P}_1$ at $\mathbb{p}$ as

$$\mathbb{N} = \sum \omega_e B_e \mathbb{v}, \tag{8.27}$$

and note that the tangent space $\mathscr{T}$ has dimension $\#\mathscr{E} - 1$. The range of Π is of the same dimension, and equation (8.26) says that the normals to the two subspaces are not orthogonal. But then it follows that each edge vector projects non-trivially onto $\mathscr{T}$: it is impossible that $\boldsymbol{\pi}_e = (\boldsymbol{\pi}_e \cdot \mathbb{N})\,\mathbb{N}$ since $\boldsymbol{\pi}_e \neq \mathbf{0}$ is orthogonal to $\mathbb{v}$ and $\mathbb{N}$ is not. Thus a collection of linearly independent elements of $(\boldsymbol{\pi}_e)_{e\in\mathscr{E}}$ projects onto a basis for $\mathscr{T}$. The same collection remains linearly independent and spanning in a neighborhood of $\mathbb{p}$ on $\mathfrak{P}_1$, so that their integral curves form a coordinate system. □

It is worth noting that if a path from $\mathbb{p} \in \mathfrak{P}_1$ leaves that surface then it cannot be admissible: in each open neighborhood of $\mathbb{p}$ all points not on $\mathfrak{P}_1$ have Π of full rank, and no prestress, so there can be no admissible velocity. This gives another interpretation of the second-order stress test (4.17): it ensures that under these circumstances no admissible path can leave the manifold, while on the manifold lengths must change on any path.

The previous result gives a convenient way of generating new stable placements from a given one. It is easy to construct a path on the $\mathfrak{P}_1$ manifold by simultaneously shortening and lengthening two or more edges in order to remain on the manifold. Moreover, given the continuity of the null-spaces, the set of stable placements is open, so that the process does not abruptly result in an unstable position. This leads (*cf* [66]), in the case in which two edges are modified at one time, to differential equations

$$\sum_e \omega_e B_e \mathbb{v} \cdot \dot{\mathbb{q}} = 0 \tag{8.28}$$

$$\boldsymbol{\pi}_e \cdot \dot{\mathbb{q}} = 0 \quad \text{for all } e \text{ but two} \tag{8.29}$$

$$\boldsymbol{\pi}_{e_o} \cdot \dot{\mathbb{q}} = 1 \quad \text{for a chosen } e_o. \tag{8.30}$$

These can be solved numerically: at each time-step the placement must be corrected to ensure precise placement upon the manifold, but this is easy to do, and the results enable construction of a sequence of stable placements. An illustration such a process, for T-3 with a fixed base, is in figure 5.

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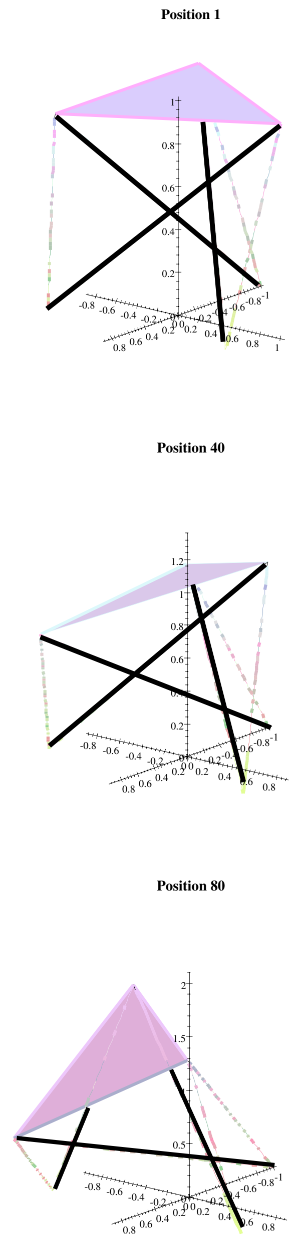

Figure 5: The Starting Placement and Two Subsequent Ones in a Numerical Solution

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