21-341 **Linear Algebra I** **Fall 2003**

Review Problems for Test 1

1. Let $\mathbb{F} = \mathbb{R}$ and $V = \mathbb{R}^4$. Determine whether or not the list of vectors is linearly independent.

   (a) $< 1, 1, 2, 1 >, \quad < 1, 1, 2, 2 >$

   (b) $< 1, -1, 1, 0 >, \quad < 1, 2, 3, 4 >, \quad < 1, 2, 3, 1 >, < -1, 1, -1, 1 >,$
   $< 1, 0, 1, 0 >$

   (c) $< 1, -1, 1, 1 >, \quad < 0, 0, 0, 0 >, < 1, 3, 2, 1 >$

   (d) $< 1, 2, -1, 0 >, \quad < 2, 1, 0, 1 >, \ < 4, 5, -2, 4 >$

2. Let $\mathbb{F} = \mathbb{R}$ and $V = \mathbb{R}^5$. Find a basis for the solution set of the equation $x_1 + 2x_2 - x_3 + x_4 - 2x_5 = 0$.

3. Prove or Disprove: Let V be a finitely generated vector space and let S, T, R be subspaces of V. Then $S + (T \cap R) = (S + T) \cap (S + R)$.

4. Let $\mathbb{F} = \mathbb{R}$ and let $n \in \mathbb{Z}^+$ be given. Let $P_n(\mathbb{R})$ denote the vector space of all real polynomials of the degree $\leq n$. Assume that $f_0, f_1, \ldots f_n \in P_n(\mathbb{R})$ satisfy $f_0(\pi) = f_1(\pi) = f_2(\pi) = \ldots = f_n(\pi) = 0$. Show that the list $f_0, f_1, f_2, \ldots, f_n$ is linearly dependent.

5. Let $\mathbb{F}$ be a field and V, W be vector spaces over $\mathbb{F}$. Let $L : V \to W$ be a mapping such that $L(u + v) = L(u) + L(v)$ and $L(\lambda u) = \lambda L(u)$ for all $u, v \in V, \ \lambda \in \mathbb{F}$. Let

   $$T = \{L(u) : u \in V\}.$$

   Show that T is a subspace of W.

6. Let $\mathbb{F} = \mathbb{R}$ and let $P_3(\mathbb{R})$ be the vector space of all real polynomials of degree ≤ 3. Let $S = \{f \in \mathbb{P}_3(\mathbb{R}) : f(1) = 2\int_0^1 f(x)dx\}$.

   Show that S is a subspace of $P_3(\mathbb{R})$ and find a basis for S.

7. Let $\mathbb{F} = \mathbb{R}$ and $A = \begin{pmatrix} 1 & 2 & 1 & -1 \\ 1 & 5 & 2 & 1 \\ -2 & 5 & 5 & 4 \end{pmatrix}$.

   Find a 3×2 matrix A' such that A' is row equivalent to A, all zero rows of A' are below the nonzero rows, and the nonzero rows of A' are in echelon form.

8. Let V be an eight dimensional vector space and let S, T be subspaces of V with $\dim(S) = 5$ and $\dim(T) = 6$. What is the smallest possible dimension of $S \cap T$.

9. Prove or Disprove: Let S, T, R be subspaces of a finitely generated vector space V. If $S + T = S + R$ then $T = R$.

10. In this problem, we will write complex numbers in the form $a+bi$ where $a, b \in \mathbb{R}$ and $i^2 = -1$; for such a number we write

$$Re(a + bi) = a, \qquad Im(a + bi) = b.$$

Let $V = \mathbb{C}^4$.

(a) Assume that $\mathbb{F} = \mathbb{R}$. Which of the following are subspaces?

i. $\{< z_1, z_2, z_3, z_4 >: \ z_1 + z_4 = 0\}$

ii. $\{< z_1, z_2, z_3, z_4 >: \ z_1 + z_4 = 1\}$

iii. $\{< z_1, z_2, z_3, z_4 >: \ z_1 + z_4 = z_2\}$

iv. $\{< z_1, z_2, z_3, z_4 : \ Re(z_1) = 0\}$

v. $\{< z_1, x_2, z_3, z_4 >: \ Im(z_1) = 0\}$

vi. $\{< z_1, z_2, z_3, z_4 >: \ z_1 z_2 = 0\}$

(b) How would the answers to part (a) change if the field were changed from $\mathbb{R}$ to $\mathbb{C}$.

11. Let $\mathbb{F} = \mathbb{R}$ and $V = \mathcal{F}(\mathbb{R})$ the set of all real-valued functions $f : \mathbb{R} \to \mathbb{R}$. Let $f_1(x) = e^x, \ \ f_2(x) = e^{2x}, \ \ f_3(x) = e^{3x}$. Determine whether or not f_1, f_2, f_3 are linearly independent.