Supplementary Problems for Assignment 4

Let $\mathbb{F} = \mathbb{R}$ and let $\mathbb{R}^{3\times 3}$ denote the set of all real 3×3 matrices. For each $u, v \in \mathbb{R}^3$ define $u \otimes v \in \mathbb{R}^{3\times 3}$ by

$$(u \otimes v)_{ij} = u_i v_j \quad \text{for all } i, j = 1, 2, 3.$$

1. (a) Let $u, v \in \mathbb{R}^3$ be given. Show that rank $(u \otimes v) \leq 1$.
   (b) Let $A \in \mathbb{R}^{3\times 3}$ be given and assume that rank$(A) \leq 1$. Show that there exist $w, z \in \mathbb{R}^3$ such that $A = w \otimes z$.

2. For each $a, b \in \mathbb{R}^3$ define $a \times b \in \mathbb{R}^3$ by

$$\langle a_1, a_2, a_3 \rangle \times \langle b_1, b_2, b_3 \rangle = \langle a_2 b_3 - a_3 b_2,\ a_3 b_1 - a_1 b_3, a_1 b_2 - a_2 b_1 \rangle$$

   Find a linear transformation $L : \mathbb{R}^{3\times 3} \to \mathbb{R}^3$ such that

$$L(u \otimes v) = u \times v \quad \forall u, v \in \mathbb{R}^3.$$

3. Let $a, b \in \mathbb{R}^3$ be given. Find a “nice” formula for $\det(I + a \otimes b)$.