Eigenvalues & Eigenvectors (Part I)

Let $\mathbb{F}$ be a field, V be a vector space over $\mathbb{F}$ and let $T \in L(V, V)$ be given.

Definition 1: Let $\lambda \in \mathbb{F}$ be given. We say that λ is an eigenvalue for T if there exists $v \in V \backslash \{0\}$ such that

$$Tv = \lambda v; \tag{1}$$

any $v \in V \backslash \{0\}$ that satisfies (1) is called an eigenvector for T corresponding to the eigenvalue λ. If λ is an eigenvalue for T then $\mathcal{N}(T - \lambda \mathbb{1})$ is called the eigenspace corresponding to the eigenvalue λ.

Definition 2: Let $n \in \mathbb{N}^+$, $\lambda \in \mathbb{F}$, and $A \in \mathbb{F}^{n \times n}$ be given. (Recall that $\mathbb{F}^{n \times n}$ is the set of all $n \times n$ matrices with entries from $\mathbb{F}$.) We say that λ is an eigenvalue for A if there is a column vector $x \in \mathbb{F}^n \backslash \{0\}$ such that

$$Ax = \lambda x; \tag{2}$$

any column vector $x \in \mathbb{F}^n \backslash \{0\}$ satisfying (2) is called an eigenvector for A corresponding to the eigenvalue λ. If λ is an eigenvalue of A, the subspace $\mathcal{N}(A - \lambda I)$ is called the eigenspace corresponding to the eigenvalue λ.

Definition 3: Assume that V is finite dimensional. We say that T is diagonalizable if there is a basis for V such that the matrix for T relative to this basis is diagonal.

Definition 4: Let $n \in \mathbb{N}^+$ and $A \in \mathbb{F}^{n \times n}$ be given. We say that A is diagonalizable if there is an invertible matrix $S \in \mathbb{F}^{n \times n}$ such that $S^{-1}AS$ is diagonal.

Proposition 1: Assume that V is finite dimensional and let A be the matrix for T relative to any basis for V. Let $\lambda \in \mathbb{F}$ be given. Then λ is an eigenvalue for T if and only if it is an eigenvalue of A.

Proposition 2: Let $v_1, v_2, \ldots, v_k$ be eigenvectors for T corresponding to distinct eigenvalues. Then the list $v_1, v_2, \ldots, v_k$ is linearly independent.

Proposition 5: Assume that V is finite dimensional and let $\lambda \in \mathbb{F}$ be given. Then λ is an eigenvalue for T if and only if

$$\det(T - \lambda \mathbb{1}) = 0. \tag{3}$$

Definition 5: Assume that V is finite dimensional. The polynomial $h : \mathbb{F} \to \mathbb{F}$ defined by

$$h(z) = \det(z\mathbb{1} - T) \quad \forall z \in \mathbb{F} \tag{4}$$

is called the characteristic polynomial for T.

Proposition 6: Let $n \in \mathbb{Z}^+,\ A \in \mathbb{F}^{n\times n}$, and $\lambda \in \mathbb{F}$ be given. Then λ is an eigenvalue for A if and only if

$$\det(A - \lambda I) = 0 \tag{5}$$

Definition 6: Let $n \in \mathbb{Z}^+$ and $A \in \mathbb{F}^{n\times n}$ be given. The polynomial $h : \mathbb{F} \to \mathbb{F}$ defined by

$$h(z) = \det(zI - A) \quad \forall z \in \mathbb{F} \tag{6}$$

is called the characteristic polynomial for A.

Theorem 1: Let $n \in \mathbb{Z}^+$ be given and assume that $\dim V = n$. If T has n distinct eigenvalues then there is a basis for V consisting solely of eigenvectors for T.

Theorem 2: Assume that $\mathbb{F} = \mathbb{R}$, V is finite dimensional and that $(\cdot,\cdot) : V \times V \to \mathbb{R}$ is an inner product. Assume further that

$$(Tx, y) = (x, Ty) \quad \forall x, y \in V.$$

Then there is an orthonormal basis for V consisting solely of eigenvectors for T.

Theorem 3: Let $\mathbb{F} = \mathbb{R},\ n \in \mathbb{Z}^+$, and $A \in \mathbb{R}^{n\times n}$ be given. Assume that

$${}^tA = A.$$

Then there is an orthonormal basis for $\mathbb{R}^n$ consisting solely of eigenvectors for A.