Supplementary Problems for Assignment 5

1. Let $\mathbb{F}$ be a field and let $T \in L(\mathbb{F}^4, \mathbb{F}^2)$ be given such that

$$\mathcal{N}(T) = \{< x_1, x_2, x_3, x_4 >\in \mathbb{F}^4 : x_1 = x_2 \text{ and } x_3 = x_4\}.$$

   Show that T is surjective.

2. Let $\mathbb{F}$ be a field. Show that there is no $T \in L(\mathbb{F}^5, \mathbb{F}^2)$ such that

$$\mathcal{N}(T) = \{< x_1, x_2, x_3, x_4, x_5 >\in \mathbb{F}^5 : x_1 = x_2 \text{ and } x_3 = x_4 = x_5\}.$$

3. Let $\mathbb{F} = \mathbb{Z}_2$ and $V = \mathbb{F}^2$. Find linear transformations $S, T \in L(V, V)$ such that

$$ST - TS = \mathbb{1}.$$

4. Let $\mathbb{F} = \mathbb{R}$ and $V = \mathbb{R}^2$. do there exist linear transformations $S, T \in L(V, V)$ such that $ST - TS = \mathbb{1}$.

5. Prove or Disprove: Let $\mathbb{F}$ be a field, V be a vector space over $\mathbb{F}$ and $S, T \in L(V, V)$ be given. Then

$$\dim(\mathcal{N}(ST)) \leq \dim(\mathcal{N}(S)) + \dim(\mathcal{N}(T)).$$