Department of Mathematical Sciences
Carnegie Mellon University

21-476 **Ordinary Differential Equations** **Fall 1997**

Assignment 3

Due on Friday, October 3

1. Assume that $g : \mathbb{R}^3 \to \mathbb{R}$ is continuously differentiable and that

$$g(t, y, 0) = 0 \qquad \text{for all } t, y \in \mathbb{R}.$$

Show that every solution of the second order scalar equation

$$\ddot{u}(t) = g(t, u(t), \dot{u}(t)) \tag{1}$$

is monotonic.

2. Let $(\alpha, \beta) \in \mathbb{R}^2$ be given and let x be the unique noncontinuable solution of

$$\begin{cases} \dot{x}_1(t) = e^{-t} x_1(t) + e^{-2t} x_2(t) \\ \dot{x}_2(t) = -e^{-2t} x_1(t) + e^{-t} x_2(t) \\ x_1(0) = \alpha, \; x_2(0) = \beta. \end{cases} \tag{2}$$

Show that $[0, \infty) \subset \mathrm{Dom}(x)$ and that x is bounded on $[0, \infty)$. What can you say about the behavior of $||x(t)||_2$ as $t \to \infty$?

3. Let $\epsilon > 0$ be given and consider the Van der Pol equation

$$\ddot{u}(t) + \epsilon(u(t)^2 - 1)\dot{u}(t) + u(t) = 0. \tag{3}$$

If we put $x_1 = u, \; x_2 = \dot{u}$, then (3) can be rewritten as the system

$$\begin{cases} \dot{x}_1(t) = x_2(t) \\ \dot{x}_2(t) = -x_1(t) + \epsilon(1 - x_1(t)^2)x_2(t). \end{cases} \tag{4}$$

Let x be a noncontinuable solution of (4) with $0 \in \mathrm{Dom}(x)$. Show that $[0, \infty) \subset \mathrm{Dom}(x)$.

4*. Assume that $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous, has the uniqueness property, and satisfies

$$f(t+1, y) = f(t, y) \qquad \forall t, y \in \mathbb{R}.$$

Assume further that the differential equation

$$\dot{x}(t) = f(t, x(t)) \tag{5}$$

has a solution x^* such that $[0, \infty) \subset \mathrm{Dom}(x^*)$ and x^* is bounded on $[0, \infty)$. Prove that (5) has a periodic solution x with period 1.