Assignment 2

Due on Friday, October 1

1. Let $x_n = \frac{n}{n+1}\left[1+\cos\left(\frac{n\pi}{2}\right)\right]$ for all $n \in \mathbb{N}$. Find all cluster points of $\{x_n\}_{n=1}^{\infty}$ and find $\limsup_{n\to\infty} x_n$ and $\liminf_{n\to\infty} x_n$.

2. Show that the sequence defined recursively by

$$x_1 = \sqrt{2}, \quad x_{n+1} = \sqrt{2+\sqrt{x_n}}$$

for all $n \in \mathbb{N}$ is convergent.

3.* Consider the sequence $\{x_n\}_{n+1}^{\infty}$ defined recursively by $x_1 = 2$, $x_{n+1} = \frac{x_n}{2} + \frac{1}{x_n}$ for all $n \in \mathbb{N}$.

   (a) Show that $\{x_n\}_{n=1}^{\infty}$ is convergent.

   (b) Find $\lim_{n\to\infty} x_n$.

4. Show that every sequence has a monotonic subsequence.

5.* Let $\{x_n\}_{n=1}^{\infty}$, $\{y_n\}_{n=1}^{\infty}$, $\{z_n\}_{n=1}^{\infty}$ be sequences and let $l \in \mathbb{R}$ be given. Assume that $x_n \to l$ as $n \to \infty$, $z_n \to l$ as $n \to \infty$ and that

$$x_n \le y_n \le z_n \quad \text{for all} \quad n \in \mathbb{N}.$$

Show that $y_n \to l$ as $n \to \infty$.

6.* Let $\{x_n\}_{n=1}^{\infty}$ be a sequence and assume that every $l \in (0,1]$ is a cluster point of $\{x_n\}_{n=1}^{\infty}$. Show that 0 is a cluster point of $\{x_n\}_{n=1}^{\infty}$.

7.* Show that there is a sequence $\{x_n\}_{n=1}^{\infty}$ such that every real number is a cluster point of $\{x_n\}_{n=1}^{\infty}$.

8. Give an example of two bounded sequences $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ such that neither sequence is convergent,

$$\limsup_{n\to\infty}(x_n + y_n) = \left(\limsup_{n\to\infty}\, x_n\right) + \left(\limsup_{n\to\infty}\, y_n\right),$$

but

$$\limsup_{n\to\infty}(x_n y_n) \neq \left(\limsup_{n\to\infty}\, x_n\right)\left(\limsup_{n\to\infty}\, y_n\right)$$

*Problems marked with an asterisk should be written up and handed in.