Solutions to Assignment 6

3. (a) Let $x \in \mathbb{R}$ be given. Then for each $h \in \mathbb{R}\backslash\{0\}$ we may choose C_h between x and $x+h$ and C_{-h} between x and $x-h$ such that

$$\begin{aligned} f(x+h) &= f(x) + f'(x)h + \frac{1}{2} f''(C_h)\, h^2 \\ f(x-h) &= f(x) - f'(x)h + \frac{1}{2} f''(C_{-h})\, h^2, \end{aligned}$$

by virtue of Taylor's Theorem. It follows that $\forall h \in \mathbb{R}\backslash\{0\}$ we have

$$\frac{f(x+h)+f(x-h)-2f(x)}{h^2} = \frac{1}{2} f''(C_h) + \frac{1}{2} f''(C_{-h}).$$

Notice that $C_h \to x$ and $C_{-h} \to x$ as $h \to 0$. Since f'' is continuous at x, we have

$$\lim_{h\to 0} f''(C_h) = f''(x), \ \lim_{h\to 0} f''(C_{-h}) = f''(x).$$

It follows that

$$\lim_{h\to 0} \frac{f(x+h)+f(x-h)-2f(x)}{h^2} = f''(x).$$

(b) Define $g : \mathbb{R} \to \mathbb{R}$ by

$$g(x) = \begin{cases} 1 & \forall x > 0 \\ 0 & x = 0 \\ -1 & x < 0, \end{cases}$$

and notice that g is not continuous at 0, so that g certainly does not have a second derivative at 0. On the other hand, $g(h) + g(-h) - 2g(0) = 0$ for all $h \in \mathbb{R}$, so that

$$\lim_{h\to 0} \frac{g(h)+g(-h)-2g(0)}{h^2} = 0.$$

5. By Taylor's Theorem, we may choose $C_1 \in (0,1)$ and $C_{-1} \in (-1,0)$ so that

$$\begin{aligned} f(1) &= f(0) + f'(0) + \frac{1}{2}f''(0) + \frac{1}{6}f'''(C_1) \\ f(-1) &= f(0) - f'(0) + \frac{1}{2}f''(0) - \frac{1}{6}f'''(C_{-1}). \end{aligned}$$

In particular, we have

$$\begin{aligned} 1 &= \frac{1}{2}f''(0) + \frac{1}{6}f'''(C_1), \\ 0 &= \frac{1}{2}f'''(0) - \frac{1}{6}f'''(C_{-1}). \end{aligned}$$

Subtracting these two equations gives $1 = \frac{1}{6}f'''(C_1) + \frac{1}{6}f'''(C_{-1})$, i.e., $f'''(C_1) + f'''(C_{-1}) = 6$. It follows that $f'''(C_1) \geq 3$ or $f'''(C_{-1}) \geq 3$.

6. Suppose that f is not the zero function. Then we may choose $x_0 \in (a,b)$ such that $f(x_0)^2 > 0$. Since f^2 is continuous we may choose $\delta > 0$ such that $(x_0 - \delta, x_0 + \delta) \subset (a,b)$ and that

$$\left| f(x)^2 - f(x_0)^2 \right| < \frac{1}{2}f(x_0)^2 \ \forall x \in (x_0 - \delta, x_0 + \delta).$$

It follows that

$$f(x)^2 \geq \frac{1}{2}f(x_0)^2 \ \forall x \in (x_0 - \delta, x_0 + \delta).$$

Therefore, we have

$$\begin{aligned} \int_a^b f^2 &= \int_a^{x_0 - \delta} f^2 + \int_{x_0 - \delta}^{x_0 + \delta} f^2 + \int_{x_0 - \delta}^{b} f^2 \\ &\geq 0 + 2\delta\left(\frac{1}{2}f(x_0)^2\right) + 0 = \delta f(x_0)^2 \\ &> 0. \end{aligned}$$

7. Choose $m, M \in \mathbb{R}$ such that

$$m \le f(x) \le M \text{ for all } x \in [a,b].$$

Let $\varepsilon > 0$ be given. Choose $c \in (a,b)$ such that

$$(M-m)(c-a) < \frac{\varepsilon}{2}.$$

Since f is integrable on $[c,b]$, we may choose $Q \in \mathcal{P}[c,b]$ such that

$$U(f,Q) - L(f,Q) < \frac{\varepsilon}{2}.$$

Put $P = \{a\} \cup Q$ and notice that $P \in \mathcal{P}[a,b]$. Observe also that

$$\begin{aligned} U(f,P) - L(f,P) &\le (M-m)(c-a) + U(f,Q) - L(f,Q) \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2}. \end{aligned}$$

It follows that $f \in \mathcal{R}[a,b]$.