Solutions to Assignment 1

3. Let $\epsilon > 0$ be given. Choose $N \in \mathbb{N}$ with $N > \frac{2}{\epsilon}$ and notice that $\frac{1}{N} < \frac{\epsilon}{2}$. Then, for all $n \in \mathbb{N}$ with $n \geq N$ we have

$$\begin{aligned} \left|\frac{2n}{n+1} - 2\right| &= \left|\frac{2n - 2(n+1)}{n+1}\right| \\ &= \frac{2}{n+1} \\ &< \frac{2}{n} \leq \frac{2}{N} < 2\left(\frac{\epsilon}{2}\right) = \epsilon. \end{aligned} \qquad \square$$

5. Assume that $\{x_n\}_{n=1}^{\infty}$ is bounded. Then we may choose $M > 0$ such that $|x_n| \leq M$ for all $n \in \mathbb{N}$. Let $\epsilon > 0$ be given. Since $y_n \to 0$ as $n \to \infty$ we may choose $N \in \mathbb{N}$ such that $|y_n| < \frac{\epsilon}{M}$ for all $n \in \mathbb{N}$ with $n \geq N$. It follows that for all $n \in \mathbb{N}$ with $n \geq N$ we have

$$|x_n y_n - 0| = |x_n y_n| = |x_n| \cdot |y_n| \leq M|y_n| < M\left(\frac{\epsilon}{M}\right) = \epsilon.$$

6. Notice that $l = \frac{1}{n}\sum_{k=1}^{n} l$ for all $n \in \mathbb{N}$. It follows that

$$(1) \qquad y_n - l = \frac{1}{n}\sum_{k=1}^{n}(x_k - l) \quad \forall n \in \mathbb{N}.$$

Let $\epsilon > 0$ be given. Choose $N_1 \in \mathbb{N}$ such that

$$(2) \qquad |x_k - l| < \frac{\epsilon}{2} \quad \forall k \in \mathbb{N},\ k \geq N_1.$$

Let

$$(3) \qquad M = \sum_{k=1}^{N_1} |x_k - l|$$

and choose $N_2 \in \mathbb{N}$ with

$$N_2 > \frac{2M}{\epsilon}. \tag{4}$$

Notice that

$$\frac{M}{N_2} < \frac{\epsilon}{2}. \tag{5}$$

Now put $N = \max\{N_1 + 1, N_2\}$. Then for all $n \in \mathbb{N}$ with $n \geq N$ we have

$$\begin{aligned}
|y_n - l| &= \frac{1}{n}\left|\sum_{k=1}^{n}(x_k - l)\right| \\
&\leq \frac{1}{n}\sum_{k=1}^{n}|x_k - l| \\
&\leq \frac{1}{n}\sum_{k=1}^{N_1}|x_k - l| + \frac{1}{n}\sum_{k=N_1+1}^{n}|x_k - l| \\
&\leq \frac{M}{n} + \frac{1}{n}\sum_{k=N_1+1}^{n}|x_k - l| \\
&\leq \frac{M}{N_2} + \left(\frac{n - N_1}{n}\right)\frac{\epsilon}{2} \\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \qquad \square
\end{aligned}$$

8. For each $n \in \mathbb{N}$ we may choose $r_n \in \mathbb{Q}$ with $l - \frac{1}{n} < r_n < l + \frac{1}{n}$ by virtue of density of $\mathbb{Q}$ in $\mathbb{R}$. Notice that

$$|r_n - l| < \frac{1}{n} \quad \forall n \in \mathbb{N}.$$

Let $\epsilon > 0$ be given. Choose $N \in \mathbb{N}$ with $N > \frac{1}{\epsilon}$. Then $\forall n \in \mathbb{N}$ with $n \geq N$ we have

$$|r_n - l| < \frac{1}{n} \leq \frac{1}{N} < \epsilon.$$

We conclude that $r_n \to l$ as $n \to \infty$. $\square$