Assignment 3

Due on Monday, October 11

1. Find $\text{int}(S)$ and $cl(S)$ for each of the following.

   (a) $S = \{x \in \mathbb{Q} : x^2 < 2\}$

   (b) $S = \displaystyle\bigcup_{n=1}^{\infty} \left(\frac{1}{n^2+1}, \frac{1}{n} \right)$

   (c) $S = \displaystyle\bigcup_{n=1}^{\infty} \left[n, n + \frac{1}{n} \right]$

   (d) $S = \{x \in \mathbb{R} \backslash \mathbb{Q} : 0 < x < 1\}$

2.* Use the definition of compactness to show that $S = \{0\} \bigcup \left\{ \frac{1}{n} : n \in \mathbb{N} \right\}$ is compact.

3.* Let $S \subset \mathbb{R}$. Show that if S is nonempty, closed, and bounded above then $(\sup(S)) \in S$.

4.* Prove or Disprove: For every set $S \subset \mathbb{R}$, we have

$$\text{int}(S^c) = (cl(S))^c.$$

5. Let $S \subset \mathbb{R}$. We say that S is *regularly open* if $S = \text{int}(cl(S))$. Prove or disprove each of the following.

   (a) The union of any collection of regularly open sets is regularly open.

   (b) The intersection of any finite collection of regularly open sets is regularly open.

6. Let $S \subset \mathbb{R}$. Show that $\text{int}(S)$ is open and $cl(S)$ is closed.

7. Prove or Disprove each of the following.

   (a) For all sets $S, T \subset \mathbb{R}$ we have $cl(S \bigcup T) = cl(S) \bigcup cl(T)$.

   (b) For all sets $S, T \subset \mathbb{R}$ we have $\text{int}(S \bigcup T) = \text{int}(S) \bigcup \text{int}(T)$.

8.* Let $\{S_i : i \in \mathbb{N}\}$ be a collection of nonempty closed subsets of $\mathbb{R}$ such that $S_{n+1} \subset S_n$ for all $n \in \mathbb{N}$.

(a) Show that if there exists $k \in \mathbb{N}$ such that S_k is bounded then $\bigcap_{n=1}^{\infty} S_n \neq \emptyset$.

(b) Give an example to show that the conclusion of part (a) may be false if we do not require one of the sets to be bounded.

9. Let $\{x_n\}_{n=1}^{\infty}$ be a real sequence. Show that the set of all cluster points of $\{x_n\}_{n=1}^{\infty}$ is closed.

*Problems marked with an asterisk should be written up and handed in.