I. The Real Number System

The main goal of this course is to develop the theory of real-valued functions of one real variable in a systematic and rigorous fashion. (Our approach will be designed so that generalization to more exotic types of functions should be reasonably natural.) In order to do this we need to have a clearly delineated concept of *real number*.

There are two basic approaches to discussing the real number system. One of them is to first introduce the integers, then define the rational numbers to be ratios of integers, and finally construct the real numbers by "filling in" certain gaps in the rational numbers; this process is called completion of the rationals. The other approach is to assume the existence of a set $\mathbb{R}$ (called the set of real numbers) having all of the appropriate properties. We shall adopt the latter approach because it will allow us to get to the central part of the course more quickly, and also because everyone in the class already has at least some familiarity with the real number system.

We take for granted the existence of a set $\mathbb{R}$, together with two distinguished elements 0 (zero) and 1 (one), with $0 \neq 1$, two binary operations $+$ (addition) and $\cdot$ (multiplication), from $\mathbb{R} \times \mathbb{R}$ to $\mathbb{R}$ and a distinguished subset $\mathbb{P} \subset \mathbb{R}$, satisfying axioms (F1) thru (F9), (O1) thru (O3), and (C) below. The elements of $\mathbb{R}$ are called *real numbers*; the elements of $\mathbb{P}$ are called *positive* real numbers or simply positive numbers.

Axioms (F1)-(F9) are called the *field axioms*; they imply all of the familiar rules of basic algebra. We shall use the various consequences of these axioms without explicit discussion. The *order axioms* (O1)-(O3) distinguish the positive real numbers. Using the notion of positive number, it is straightforward to define the relations "less than" and "greater than". We shall also use the basic properties of inequalities (which follow from (O1)-(O3)) without explicit discussion. The completeness axiom (C) is what distinguishes the real numbers from other ordered fields. (Any system satisfying (F1)-(F9) and (O1)-(O3) is called an *ordered field*. The rational numbers form an ordered field but they do not satisfy (C).) In contrast with the previous axioms, we shall usually be very explicit about the use of (C).

The Field Axioms

(F1) $\forall x, y \in \mathbb{R}, \quad x + y = y + x$
(F2) $\forall x, y, z \in \mathbb{R}, \ (x + y) + z = x + (y + z)$
(F3) $\forall x \in \mathbb{R}, \quad x + 0 = x$
(F4) $\forall x \in \mathbb{R}, \quad \exists y \in \mathbb{R}$ such that $x + y = 0$
(F5) $\forall x, y \in \mathbb{R}, \quad x \cdot y = y \cdot x$
(F6) $\forall x, y, z \in \mathbb{R}, \quad (x \cdot y) \cdot z = x \cdot (y \cdot z)$
(F7) $\forall x \in \mathbb{R}, \quad x \cdot 1 = x$

(F8) $\forall x \in \mathbb{R}\backslash\{0\}, \quad \exists y \in \mathbb{R}$ such that $x \cdot y = 1$
(F9) $\forall x, y, z \in \mathbb{R}, \quad x \cdot (y+z) = (x \cdot y) + (x \cdot z)$

Remark I.1: (a) It is straightforward to show that for every $x \in \mathbb{R}$ there is exactly one $y \in \mathbb{R}$ such that $x + y = 0$; it is customary to denote this y by $-x$.

(b) It is also straightforward to show that for every $x \in \mathbb{R}\backslash\{0\}$ there is exactly one $y \in \mathbb{R}$ such that $x \cdot y = 1$; it is customary to denote this y by $\frac{1}{x}$ or by x^{-1}.

Remark I.2: We define the operations of subtraction and division by

$$x - z = x + (-z) \quad \text{for all } x, z \in \mathbb{R},$$

$$\frac{x}{y} = x \cdot \left(\frac{1}{z}\right) \quad \text{for all } x \in \mathbb{R},\ z \in \mathbb{R}\backslash\{0\}.$$

Remark I.3: When there is no danger of confusion we omit the $\cdot$ and write xy in place of $x \cdot y$.

Order Axioms

(O1) $\forall x, y \in \mathbb{P}, \quad x + y \in \mathbb{P}$ and $xy \in \mathbb{P}$
(O2) $\forall x \in \mathbb{P}, \quad -x \notin \mathbb{P}$
(O3) $\forall x \in \mathbb{R}, \quad x = 0$ or $x \in \mathbb{P}$ or $-x \in \mathbb{P}$.

Remark I.4: We define the relation $<$ on $\mathbb{R}$ by $x < y \Leftrightarrow y - x \in \mathbb{P}$.
We define the additional relations $\leq$, $>$, and $\geq$ on $\mathbb{R}$ by

$$\begin{aligned} x \leq y &\Leftrightarrow x < y \text{ or } x = y \\ x > y &\Leftrightarrow y < x \\ x \geq y &\Leftrightarrow y \leq x. \end{aligned}$$

Before stating the final axiom (C) we need some definitions.

Definition I.5: Let $S \subset \mathbb{R}$ and $b \in \mathbb{R}$ be given. We say that b is

(i) an *upper bound* for S if $x \leq b$ for all $x \in S$.

(ii) a *lower bound* for S if $x \geq b$ for all $x \in S$.

Definition I.6: Let $S \subset \mathbb{R}$ and $c \in \mathbb{R}$ be given. We say that c is

(i) a *least upper bound* for S if c is an upper bound for S and $c \leq b$ for every b that is an upper bound for S.

(ii) a *greatest lower bound* for S if c is a lower bound for S and $c \geq b$ for every b that is a lower bound for S.

Remark I.7: It is easy to see that a set $S \subset \mathbb{R}$ can have at most one least upper bound and at most one greatest lower bound. We denote the least upper bound for S (if it exists) by $\sup(S)$ and the greatest lower bound for S (if it exists) by $\inf(S)$.

Definition I.8: Let $S \subset \mathbb{R}$. We say that S is

(i) *bounded above* if S has an upper bound $b \in \mathbb{R}$.

(ii) *bounded below* if S has a lower bound $b_* \in \mathbb{R}$.

(iii) *bounded* if S is bounded above and bounded below

We are now ready to state the completeness axiom.

Completeness Axiom

(C) Every nonempty subset of $\mathbb{R}$ that is bounded above has a least upper bound.

In order to discuss some important subsets of $\mathbb{R}$ (namely the natural numbers, the integers, and the rational numbers) it is convenient to introduce the idea of an inductive set.

Definition I.9: Let $S \subset \mathbb{R}$. We say that S is an *inductive set* provided that

(i) $1 \in$S, and

(ii) $\forall x \in S, \quad x+1 \in S$.

Remark I.10: It is straightforward to show that the intersection of any collection of inductive sets is an inductive set.

Definition I.II: We define the set $\mathbb{N}$ of *natural numbers* by

$$\mathbb{N} = \cap \mathcal{C},$$

where $\mathcal{C}$ denotes the collection of all inductive sets (i.e., $\mathbb{N}$ is the smallest inductive set.

Definition I.12: We define the set $\mathbb{Z}$ of *integers* by

$$\mathbb{Z} = \mathbb{N} \cup \{0\} \cup \{x \in \mathbb{R} : -x \in \mathbb{N}\}.$$

Definition I.13: We define the set $\mathbb{Q}$ of *rational numbers* by

$$\mathbb{Q} = \left\{ \frac{m}{n} : m, n \in \mathbb{Z}, \quad n \neq 0 \right\}.$$

Remark I.14: Notice that

$$\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R}.$$

The principle of *mathematical induction* follows immediately from our definition of $\mathbb{N}$ as the smallest inductive set. Due to the importance of this principle we shall state it explicitly.

Principle of Mathematical Induction: Let S be a subset of $\mathbb{N}$ and assume that

(i) $1 \in S$, and

(ii) $\forall n \in S, \quad n+1 \in S$.

Then $S = \mathbb{N}$.

Another important principle that is closely related to induction is stated below. In analysis textbooks this is often referred to as the *well-ordering principle* (for the natural natural numbers). Some people refer to it as the least number principle (and reserve the term well-ordering for something more general.)

Well-Ordering Principle (for $\mathbb{N}$): Every nonempty subset of $\mathbb{N}$ has a smallest element.

We need to briefly discuss a few issues pertaining to finite and infinite sets.* For this purpose we recall some definitions concerning functions.

Definition I.15: Let D and E be sets and consider a function $f : D \to E$. We say that f is

(i) *injective* provided that $\forall x_1, x_2 \in D$ with $x_1 \neq x_2$, we have $f(x_1) \neq f(x_2)$.

(ii) *surjective* if $\forall y \in E, \ \exists x \in D$ with $f(x) = y$.

(iii) *bijective* if it is injective and surjective.

Definition I.16: Let S be any set. We say that S is

(i) *infinite* if there is an injective function $f : \mathbb{N} \to S$.

(ii) *finite* if it is not infinite.

(iii) *countably infinite* if there exists a bijective function $f : \mathbb{N} \to S$

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*Unless stated otherwise, we assume that the axiom of choice holds.

(iv) *countable* if it is either finite or countably infinite.

(v) *uncountable* if it is not countable.

The following proposition contains a number of basic facts concerning the concepts of the previous definition.

Proposition I.17:

(i) Every subset of a finite set is finite.

(ii) Every subset of a countable set is countable.

(iii) If a set S has an infinite subset then S is infinite.

(iv) If a set S has an uncountable subset then S is uncountable.

(v) The union of any finite collection of finite sets is finite.

(6) The union of any countable collection of countable sets is countable.

Proposition I.18: The sets $\mathbb{N}$, $\mathbb{Z}$, and $\mathbb{Q}$ are countably infinite.

Proposition I.19: Let $\alpha, \beta \in \mathbb{R}$ with $\alpha < \beta$ be given and put

$$(\alpha, \beta) = \{x \in \mathbb{R} : \alpha < x < \beta\}.$$

Then (α, β) is uncountable.

It may seem intuitively clear that the natural numbers are not bounded above. However, this cannot be proved using only the field axioms and the order axioms. (Indeed there are ordered fields in which the set of natural numbers is bounded above.) For our first use of the completeness axiom, we shall prove the following basic result.

Proposition I.20: The set $\mathbb{N}$ is not bounded above.

Proof: Suppose that $\mathbb{N}$ is bounded above. Since $\mathbb{N}$ is nonempty, the completeness axiom implies that $\mathbb{N}$ has at least upper bound. Let $c = \sup(\mathbb{N})$. Then $c - 1$ is not an upper bound for $\mathbb{N}$. Therefore, we may choose $n \in \mathbb{N}$ such that $n > c - 1$. Then $n + 1 > c$ and $n + 1 \in \mathbb{N}$. This contradicts the fact that c is an upper bound for $\mathbb{N}$. ■

The so-called Archimedean Property of $\mathbb{R}$ is an immediate consequence of Proposition I.20.

Archimedian Property of $\mathbb{R}$: Let $x \in \mathbb{R}$ be given. There exists $n \in \mathbb{Z}$ such that $n > x$.

Density of $\mathbb{Q}$ in $\mathbb{R}$: Let $x, y \in \mathbb{R}$ with $x < y$ be given. Then there exists $r \in \mathbb{Q}$ with $x < r < y$.

Proof: Assume first that $x \geq 0$. By the Archimedean Property of $\mathbb{R}$ we may choose $q \in \mathbb{N}$ such that

$$q > \frac{1}{y-x}. \tag{1}$$

Since $q, y - x > 0$ we have

$$0 < \frac{1}{q} < y - x \tag{2}$$

Let $S = \{n \in \mathbb{N} : y \leq \frac{n}{q}\}$. By the Archimedean Property, S is nonempty. By the Well-Ordering Principle, S has a smallest element. Let $p = \min(S)$. Then $p \in S$, i.e.

$$y \leq \frac{p}{q} \tag{3}$$

and $p - 1 \notin S$, i.e.

$$\frac{p-q}{q} < y. \tag{4}$$

Notice that

$$\begin{aligned} x &= y + (x - y) \\ &\leq \tfrac{p}{q} + (x - y), \end{aligned} \tag{5}$$

by virtue of (3). It follows from (2) that

$$(x - y) < -\frac{1}{q} \tag{6}$$

Combining (5) and (6) we obtain

$$x < \frac{p-1}{q},$$

and the desired conclusion holds with $r = \frac{p-1}{q}$.

If $x < 0$ we may choose $n \in \mathbb{N}$ with $n > -x$. Then $n + x > 0$ and we may choose $r_* \in \mathbb{Q}$ with $n + x < r_* < n + y$ by the argument above. Notice that $r_* - n \in \mathbb{Q}$ and $x < r_* - n < y$. ■

We close this section with two propositions concerning finiteness and the natural numbers.

Proposition I.21: A nonempty set S is finite if and only if there exist $N \in \mathbb{N}$ and a bijection $f : \{n \in \mathbb{N} : 1 \leq n \leq N\} \to S$.

Proposition I.22: A set $K \subset \mathbb{N}$ is finite if and only if K is bounded above.