Some Additional Remarks on Fields

A. Ordered Fields: Let $\mathbb{F}$ be a field. By a *positive half* or *positive part* for $\mathbb{F}$, we mean a set $\mathbb{P} \subset \mathbb{F}$ satisfying (1) and (2) below.

(1) $\forall \alpha, \beta \in \mathbb{P},\ \alpha + \beta \in \mathbb{P}$ and $\alpha\beta \in \mathbb{P}$.

(2) For each $\alpha \in \mathbb{F}$ exactly one of the following three conditions holds:
$\alpha \in \mathbb{P},\ \alpha = 0,\ -\alpha \in \mathbb{P}$.

By an *ordered field* we mean a field together with a positive half. A given field may have no positive half, exactly one positive half, or more than one positive half. These possibilities are illustrated in Assignment 1.

B. The Characteristic of a Field: Let $\mathbb{F}$ be a field. Given $\alpha \in \mathbb{F}$ and $n \in \mathbb{Z}^+$, we can define $n\alpha \in \mathbb{F}$ by induction as follows: $1\alpha = 1$; given $k \in \mathbb{Z}^+$ and assuming $k\alpha$ has been defined, we put $(k+1)\alpha = k\alpha + \alpha$. Let $S = \{n \in \mathbb{Z}^+ | n1 = 0\}$. If $S = \emptyset$, we say that $\mathbb{F}$ has *characteristic zero*. If $S \neq \emptyset$, we define the *characteristic* of $\mathbb{F}$ to be the smallest element of S. It can be shown that the characteristic of a field is either zero or prime.

C. The Field $\mathbb{Z}_p$: Let $n \in \mathbb{Z}^+$ be given and let

$$\mathbb{Z}_n = \{0, 1, 2, \ldots n-1\}$$

equipped with addition and multiplication modulo n. It can be shown that $\mathbb{Z}_n$ is a field if and only if n is prime.

D. General Finite Fields: It can be shown that if $\mathbb{F}$ is a finite field then the number of elements of $\mathbb{F}$ is of the form p^n for some prime number p and some $n \in \mathbb{Z}^+$. It can be also shown that given any prime p and $n \in \mathbb{Z}^+$ there is a field having exactly p^n elements. Moreover, for any field having exactly p^n elements (where p is prime and $n \in \mathbb{Z}^+$) the characteristic is p.