

Step 1: Understand Goal

What am I looking for? What should the last sentence in my proof say? What are some equivalent ways to state what I want? Are there any complicated definitions that need to be “unwound”? Can I see any assumptions that would make this true?

Step 2: Understand Assumptions

What am I allowed to assume? What are some equivalent ways to state what I know? Are there any complicated definitions that need to be “unwound”? What problems do I face when trying to find a counterexample? Can I find any counterexamples to my conclusion when some of these assumptions are eliminated or weakened? What evidence do I have that these assumptions yield that conclusion?

Step 3: Understand Relevant Information

Can I think of any similar problems? Are there any theorems with similar conclusions? Is there a way to connect the assumptions I have to similar conclusions? What information can I obtain from the assumptions? Are there any theorems with similar assumptions?

Step 4: Form a Plan

Can I use any of the relevant information that I found? Can any of the proofs that have similar conclusions or conclusions be adapted for this circumstance? Are there any auxiliary theorems I can prove first? Can some assumptions be strengthened or added can I reach my conclusion? If so, can that proof be adapted for the assumptions I do have? Can some of my conclusions be weakened or eliminated to reach that modified conclusion? If so, can that proof be adapted for the conclusions I do have?

Step 5: Execute Plan

Is this step justified? Is this step relevant? Can I prove this fact? Am I using all the assumptions I have at my disposal?

Step 6: Reflection

Are all my steps justified? Can I prove all the claims that I made? Do all steps seem clear to my readers? Is there a way to verify the result? Did I actually achieve my goal? Did I use all my assumptions?

Step 7: Expand

Did I use all of my assumptions? Did I use the full strength of all of my assumptions? Can I weaken any of my assumptions? Did I prove more than I intended? Can I strengthen my conclusion?

Explanations

Step 1: Understand Goal

What am I looking for? What should the last sentence in my proof say?

This is a question that I personally ask myself a dozen or so times every time I write a proof. It's the most important question. What do I want to get out of this proof? What conclusion? This is the thing that should be in the forefront of your mind, and it should always be the first thing you ask when you get offtrack.

What are some equivalent ways to state what I want? Are there any complicated definitions that need to be “unwound”?

There are many equivalent ways to say the same thing in math. For example, one can say two integers are relatively prime if there is no positive integer that divides them both. But another way to say that is the greatest common divisor is 1. This is an easy observation. So easy in fact that you might not even think it's worth thinking about. But realizing the connection might mean noticing theorems that have gcd's involved that we wouldn't have normally noticed. Same for other definitions. Some definitions are complicated. Breaking them down into their components and working with those simpler elements usually gives a better opportunity to apply other results, and also shows you what is *actually* going on.

Can I see any assumptions that would make this true?

We haven't talked about the assumptions you do have, but often it's useful to think about what would be sufficient for your goal to be true. Sort of like a wish list. This can sometimes give you an approach (an approach could be to prove that you have everything on your wish list). If nothing else, it should give you a good background to understanding how some assumptions connect to your conclusion, which will help.

Step 2: Understand Assumptions

What am I allowed to assume?

Most statements have some assumptions. After understanding what your goal is, the next important task is understanding how your the things that you're allowed to assume yield your conclusion. The first task to this will be to understand how they relate. Sometimes it is obvious how they relate, but very often it is not. So you must begin by identifying your assumptions

What are some equivalent ways to state what I know? Are there any complicated definitions that need to be “unwound”?

As before with looking at alternate forms of the conclusion, we can look at alternate forms of the assumptions. Look at definitions and break them down into smaller atoms.

What problems do I face when trying to find a counterexample?

Look for a counterexample. On a homework problem, you probably won't find one (if you do, make sure it's right and tell somebody), so hopefully you'll get stuck. Which is good. You have to think about why you are getting stuck. Right now, you're not trying to write a proof. You are trying to just understand why this thing is true! With this understanding, you'll be an expert and you will have already seen a lot of the pitfalls by the time you get to the actual nuts and bolts of the proof.

Can I find any counterexamples to my conclusion when some of these assumptions are false? What evidence do I have that these assumptions yield that conclusion?

Sometimes this is not crucial to do, but certainly when you have the time to do this you should. It will give you a lot of understanding how each assumption yields the conclusion. Think of a theorem as a scale, with hypotheses on one side, and conclusions on the other. A good theorem is a balanced scale, and bad theorems (notice: I didn't say false theorems) are unbalanced. By in large the problems you'll be given in this class are balanced, meaning all assumptions are necessary, and there's not a whole lot more we can prove using those assumptions.

Lacking proper assumptions makes one side of the scale very light, and makes it unbalanced. Looking for counterexamples when you lose some of your assumptions gives you a lot of insight into the connection between the assumptions and conclusions. It gives you reasons why an assumption is necessary for your conclusion to follow. Therefore finding a counterexample can expose what an assumption “contributes” to proving the theorem.

Step 3: Understand Relevant Information

Can I think of any similar problems?

The most important thing here is the similar means whatever you want it to mean. It might mean it's about a similar topic. It might mean you wish you could apply it. It might mean you're encountering the same problems here that you had when you proved the other problem. Maybe it just means that's it's activating the same neurons, or it smells the same. Regardless, there are usually similar problems. Even when a problem is "new" there's almost always ideas or techniques from other proofs that can be used in this. Recognizing what similar means is the big hurdle though, and it comes only from wrestling with the previous two steps (understanding the problem).

Are there any theorems with similar conclusions? Is there a way to connect the assumptions I have to similar conclusions?

Here you want to think about things with the same or similar conclusions. Think about the proof techniques used in those theorems, or how you could apply those theorems to the problem at hand. As before, similar can mean a lot of different things here. As you prove more things, you'll get a better feel for what "similar conclusions" are. An important thing to know is that we recycle two things in math: results and techniques. Results are just as they sound. Oh, this theorem that I've proved says under these circumstances which I have than I get this thing which is really similar to what I want. I can use this theorem. Techniques are different though. Oh, this theorem I proved has a conclusion like the one I want and, although I can't apply the theorem direction, the proof can be altered to give me a similar result to the one I want.

What information can I obtain from the assumptions? Are there any theorems with similar assumptions?

This goes back from all the knowledge you got from step 2. You have a good idea of what your assumptions mean and why they're important. You've investigated the connections between them and what you're trying to show. Now, what theorems can I use to help me out? What information do I know follows? What have I ascertained?

Step 4: Form a Plan

How can I use the relevant information that I found? Can any of the proofs that have similar conclusions or conclusions be adapted for this circumstance?

Here, I just mean to use the stuff from step 3.

Are there any auxiliary theorems I can prove first?

An auxiliary theorem is a theorem that isn't quite what we want, but will help us along the way. Another name for an auxiliary theorem is a lemma. Call it what you want. Sometimes these things come from a bit of defeat. That is, we can't get our complete result, so we do a little bit. Other times they come from an observation. Oh, it would really help if I knew this fact. Let's prove that instead. But realize that they always come from somewhere. In text books (and unfortunately and inevitably, sometimes in class and recitations) they're often presented as from magic, they never are. They come from our exploration.

Can some assumptions be strengthened or added can I reach my conclusion? If so, can that proof be adapted for the assumptions I do have?

Okay, maybe we can't get what we want with what we know. But we might get stuck places. Let's add the bit to get us past that point and figure out the proof from there. Then, later on we will try to pick at what we added and eliminate all those extra assumptions. This helps, just like everything else, because it increases our understanding. Sure, it's not ideal. We don't have a good idea about how the assumptions we do have relate to what we want. But it does give us an idea about what would be "enough", or to borrow a more mathematical word: sufficient.

Can some of my conclusions be weakened or eliminated to reach that modified conclusion? If so, can that proof be adapted for the conclusions I do have?

So, we investigated what happens when we strengthened what we are assuming. Let's try weakening what we want. Remember the analogy from before. This is a scale, and there's two ways to unbalance it. An unbalanced scale isn't ideal, but this is part of a process to chip away to get what we want. So, maybe we can't get everything we want right away. But if we can get a part of that result, that might give us a lot of insight on how to get the rest.

Step 5: Execute Plan

Is this step justified? Is this step relevant? Can I prove this fact?

So, here's the deal. You're writing a proof. Every step should have 3 characteristics: 1, it should be backed up by a fact, and should be justified or sufficiently self-evident; 2, it should be relevant, and should be heading from the direction the last step was at and heading toward the conclusion we want; 3, you should completely understand why everything you said was true.

Am I using all the assumptions I have at my disposal?

This is a little trick, that maybe doesn't belong in this step, but it's too important not to say. Often times when you get stuck you have to make sure that you're using all you can use. If you notice that you've never actually use something and you're stuck, think about how to use it.

Step 6: Reflection

Are all my steps justified? Can I prove all the claims that I made?

Beating a dead horse here. You need to make sure you can prove every claim you make, and make sure every step you take is justified. Otherwise, what's the point?

Do all steps seem clear to my readers?

Proofs are supposed to be convincing. If you have not convinced me (who for the homework is your target audience) your proof is essentially worthless. The easiest way to get me to be confused is not to be clear. There are a few ways to fall into this trap. One is to make a very large leap. Maybe something seems clear to you when you're writing it, and you're in the zone, and you've just been working on the problems for hours. It won't be clear to me. I'll say "Too fast!!" and mark red on your paper.

Another way is to go off track. You might say something that I just don't trust is relevant. This is another reason to break things down into smaller claims and auxiliary theorems (lemmas). Then I can see the immediate goal, and not be confused when we don't seem to be getting close to the distant one.

The last way, and probably the most common for people starting (ie. you), is to be nebulous and not seem like you know what you're talking about. This is normally people who are trying (oh trying) to get one past the grader, and want to look like they're the "Too fast!!" people. Really though, they don't know. They don't see it. They don't have a great understanding, they're just trying to get one past the reader. Something for you to know: this is extremely obvious to us. It's hard for you to know that for sure, because most of the proofs you are presented are done moderately well, and no attempt at subversion is attempted. It is noticeable though. Credibility is a bad thing to lose too. The more credibility you have, the more "Too fast!!" 's you can get away with.

A good way to make sure your work is clear is write a fairly nice draft (although probably not your final draft) of your proof. Then don't even think about it for at least 24 hours. Then, the next day you should read it. Does it still make sense to you? If it doesn't to you then it certainly won't make sense to anyone else. The more time you can put between writing it and reading it the better. I look back at some of the proofs I wrote years ago when I was in a similar position to you and cringe. "Oh my, my teacher must have thought I was the stupidest person ever." So reading over your work a little after you write it will save yourself the embarrassment I have for myself.

Is there a way to verify the result?

If you're solving an equation, make sure the solutions work. If you're taking an antiderivative or solving a differential equation, check your answer. For example if I tell you to factor a very large number that could be difficult. You might spend awhile on it. Aren't you going to multiply the numbers together after you're done to make sure you're right before submitting your answer? Verifying an answer is usually a much easier task than solving the question. So if it's possible to verify it, you should.

Did I actually achieve my goal?

Not a whole lot to say. Make sure you achieved your goal. Otherwise you did the wrong problem. If the question is asking for a rate, don't give a volume.

Did I use all my assumptions?

In this class, you're generally given assumptions and asked to derive a conclusion. Cool. But what happens if you have stuff left over? It's like IKEA. You bought a dresser, and there's just a bunch of bolts left over at the end. Is that good? Probably not, because IKEA thought about all the stuff you'd need and

gave you just enough. The same for us. Mostly, we think about what you need and don't give you a lot extra. Sometimes we might, but more often than not, if you have an extra assumption in the class, then you did something wrong. Look back. Remember step 2 when you tried to develop a counterexample from the theorem without that assumption? Well, does your proof address that issue at all?

In real life however, you might be building your own dresser from your own materials. There's no instruction manual. You might have a lot of extra stuff left over. Maybe you could have made a better dresser out of the stuff you had? Maybe not, and you should note for the next person you don't actually need all the materials you had to build this thing. That's where the next step comes in.

Step 7: Expand

Did I use all of my assumptions? Did I use the full strength of all of my assumptions? Can I weaken any of my assumptions?

So, by this stage you have a proof, and you're pretty sure you're right. Beyond turning it in to some authority figure to make sure (yes, even researchers do this) there's nothing else you can really do to make sure the proof is correct. So we just have to go off the assumption that it is. So, now we're on the next stage which is really improving upon your result. Did you use all the assumptions? Did I use their full strength? If you didn't, maybe you messed up. But maybe you didn't. Maybe you proved something stronger than you wanted. In which case, now is the time to say, "Whoa, hold on. I know I only wanted this result, but I got this stronger thing! Isn't that cool?"

Did I prove more than I intended? Can I strengthen my conclusion?

So, maybe you didn't use any less of your full set of assumptions, but maybe you could have went a little further in your proof and got even a cooler result. It's worth considering. Sometimes (we'll see this in class) the proof we have written is already a proof of a strictly stronger statement, it just takes an observation at the end to make it official. We want millage out of these proofs. Remember the scale. If our conclusion could be stronger, than our scale is unbalanced and we can try to make it better.