Summary of Day 20

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1 Objectives

- Defined and look at properties of orthogonal matrices.

2 Summary

- Recall:

  Theorem The columns of $m \times n$ matrix Q form an orthonormal set if and only if $Q^T Q = I_n$.

- The above leads to an interesting idea. Recall that we can view the columns of a matrix as the result of where the standard basis gets sent to under viewing that matrix as a linear transformation. An interesting kind of linear transformation is one in which the standard basis gets sent to an orthonormal set (as the standard basis itself is orthonormal, the hope is that this kind of transformation will preserve a lot of geometric structure).

  We call a square $n \times n$ matrix whose columns form an orthonormal set a **orthogonal matrix**.

  Theorem Q is orthogonal if and only if $Q^T = Q^{-1}$.

  Example Rotation matrices are orthogonal

- Geometrically, orthogonal matrices distort space is very nice ways. Namely, they preserve lengths. This property is called being an **isometry**.

  Theorem If Q is a $n \times n$ matrix then TFAE:

  1. Q is orthogonal.
  2. $(Qx) \cdot (Qy) = x\dot{y}$ for every $x, y \in \mathbb{R}^n$
  3. $\|Qx\| = \|x\|$ for every $x \in \mathbb{R}^n$ (this is the property that says Q is an isometry).

  Proof. This proof is omitted. It is theorem 5.6 in the book.

Example

- Let's change gears slightly and look at orthogonal properties of subspaces. Let us recall that there are two very important subspaces associated with a $m \times n$ matrix A:
  - The column space: $\mathrm{col}(A) \subseteq \mathbb{R}^m$. This is the range of A viewed as a linear transformation.
  - The null space: $\mathrm{null}(A) \subseteq \mathbb{R}^n$. This is the kernel (the vectors sent to $\mathbf{0}$) of A viewed as a linear transformation.

  There are two other subspaces which are really these in disguise:
  - The row space: $\mathrm{row}(A) \subseteq \mathbb{R}^n$. This is $\mathrm{col}(A^T)$.
  - The **left null space**: This is $\{ x \in \mathbb{R}^m \mid xA = \mathbf{0} \}$. It is also $\mathrm{null}(A^T)$ (can you see why?).
- The row space and null space are the same type of object; they're both subspaces of $\mathbb{R}^n$. We expect there to be some type of relationship between the two, and there indeed is.

  We already know that the dimensions of the two add up to n by the rank-nullity theorem (and the fact that the dimension of the row space is always equal to the dimension of then null space). There is actually a more fundamental relationship.
- If W is a subspace of $\mathbb{R}^n$ then the **orthogonal complement** of W, $W^\perp$ (pronounced 'W perp') is the set of vectors which are orthogonal to vectors in W:

$$W^\perp = \{ \mathbf{x} \in \mathbb{R}^n \mid \forall \mathbf{w} \in W \,\boldsymbol{.}\, \mathbf{x} \cdot \mathbf{w} = 0 \}$$

  Theorem Let W be a subspace of $\mathbb{R}^n$. Then:
  - $W^\perp$ is a subspace of $\mathbb{R}^n$.
  - $\left(W^\perp\right)^\perp = W$.
  - $W \cap W^\perp = \emptyset$ (that is, the only set in common of each is $\mathbf{0}$).
  - $W = \mathrm{span}(\mathbf{w}_1, \ldots, \mathbf{w}_k)$ then $\mathbf{v} \in W^\perp$ if and only if $\mathbf{v} \cdot \mathbf{w_i}$ for all $1 \leq i \leq k$.

    Proof.

    □

Example

- We can now finally give the fundamental relationship between the null space and the row space of a $m \times n$ matrix A (and therefore, the relationship between the column space and left null space as they are the same thing but for the matrix A^T).

  Theorem If A is a $m \times n$ matrix then:

  $$(\text{row}(A))^{\perp} = \text{null}(A)$$

  Proof.

  $\square$

Example

- Let W be the subspace of $\mathbb{R}^5$ spanned by:

  $$\mathbf{w_1} = \begin{pmatrix} 1 \\ -3 \\ 5 \\ 0 \\ 5 \end{pmatrix} \qquad \mathbf{w_2} = \begin{pmatrix} -1 \\ 1 \\ 2 \\ -2 \\ 3 \end{pmatrix} \qquad \mathbf{w_3} = \begin{pmatrix} 0 \\ -1 \\ 4 \\ -1 \\ 5 \end{pmatrix}$$

  Get a basis for $W^{\perp}$.

- In another class you might have explored the idea of a projection of one vector onto another. Let us explore that idea

- We can see the projection of $\mathbf{v}$ onto $\mathbf{u}$ is given by:

$$\operatorname{proj}_{\mathbf{u}} \mathbf{v} = \left(\frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\|^2} \right) \mathbf{u}$$

- We can extent this idea to the projection of a vector onto a space.

  Let $\mathbf{v}$ be a vector of $\mathbb{R}^n$ and W a subspace and $\{\mathbf{u}_1, \ldots, \mathbf{u}_k\}$ is an orthonormal basis for W then we say the **orthogonal projection of v onto** W is defined to be:

$$\operatorname{proj}_W(\mathbf{v}) = \operatorname{proj}_{\mathbf{u_1}}(\mathbf{v}) + \cdots + \operatorname{proj}_{\mathbf{u_k}}(\mathbf{v})$$

  A worry, of course, is that this might depend on the basis. We will see that it does not.