Summary of Day 1

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1 Objectives

- Recognize linear equations.
- Define a system of linear equations.
- Build geometric intuition for a solution for a system of linear equations (in $\mathbb{R}^2$ and $\mathbb{R}^3$ anyway).
- Solve a system back back substitution.
- Express a system as an augmented matrix.

2 Summary

- An equation is **linear** if it is of the form

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$$

we call the a_i the **coefficents** and b the **constant**.

Example

- The following is a linear equation:

$$2x + 3y = 1$$

The coefficients are 2 and 3 and the constant is 1.

- The following is also a linear equation:

$$\sqrt{2}x + \pi/4y - \sin(\pi/5)z = z$$

The coefficents are $\sqrt{2}, \pi/4$ and $\sin(\pi/5)$. Note: the coefficients can be any real numbers.

- The following is not a linear equation:

$$2x^2 + \sqrt{y} + \sin(z) = 1$$

The variables x, y and z all non-linear since it cannot be put in the above form.

- The following appears not to be a linear equation:

$$3x + \sin^2(x) - 2y = -\cos^2(y)$$

But, if you do so algebra and use the trigonometric identity $\sin^2(x) + \cos^2(x) = 1$ it is:

$$3x - 2y = -1$$

Remark We slightly contrast the notion of a linear expression with a linear function. We will see linear functions later in the course; they are function which have the property $f(x + y) = f(x) + f(y)$ and $f(c \cdot x) = c \cdot f(x)$. If you take a linear equation and solve it for one variable you do not necessarily get a linear function (can you see why?).

- A **vector** (in the vector space $\mathbb{R}^n$) is an n-tuple of real numbers (i.e. a list of n real numbers). We will use bold face for vectors, so $\mathbf{v} \in \mathbb{R}^n$. In blackboard notation, we will write a vector $\bar{v}$.

  To give the **coordinates** (or **components**–the ordered elements from the n-tuple) of vector we either write $\mathbf{v}$ as a **column vector** or a **row vector** which we write as:

$$\mathbf{v} = \begin{pmatrix} s_1 \\ s_2 \\ \vdots \\ s_n \end{pmatrix} \qquad\qquad \mathbf{v} = [s_1, s_2, \ldots, s_n]$$

  respectively.

- A **solution** to a linear equation of n variables (as expressed above) is a vector $\mathbf{v}$ of $\mathbb{R}^n$, $\mathbf{v} = [s_1, \ldots, s_n]$ where

$$a_1 s_1 + \cdots a_n s_n = b$$

  i.e., when you replace the variables with the corresponding components of vector then the two sides of the equation are actually equal.

  Example The vectors $\mathbf{v} = [0, 3]$ and $\mathbf{w} = [2, 2]$ are solutions to $x + 2y = 6$

- A **system of linear equations** is a finite set of linear equations, possibly with overlapping variables. A **solution** to a system is a solution to each of the equations in the system. The **solution set** for a system of equations is the set of all vectors which are solutions to the system.

  Example The following is a system of equations:

$$\begin{aligned} 2x + 3y + z &= 1 \\ x + 2y \quad &= 0 \end{aligned}$$

  The vector $\mathbf{v} = [0, 0, 1]$ is a solution, but it is not the entire set of solutions; do you see any other solutions to the system?

- Geometrically, in if you have two linear equations these can be visualized as lines in the plane $\mathbb{R}^2$. The solution set of this system is the points of intersection. Similarly for $\mathbb{R}^3$, except the equations may also be planes.

- Two systems are called **equivalent** if they have the same solution set.

- A system is **consistent** if it has a solution (i.e. the solution set is nonempty). Otherwise, the system is **inconsistent**.

  Theorem Every system that is consistent either has one solution or infinitely many solutions.

  Example The above system is consistent because it has the solution $[0, 0, 1]$. The following system is inconsistent:

$$\begin{aligned} 2x + 3y + z &= 1 \\ 2x + 3y + z &= 0 \end{aligned}$$

- **Back substitution** is an algorithm for which you can find the solution to particular systems of equations (see the Algorithms section).

  Example The procedure for back substitution requires on a particular form of a system; we can do it when the system is 'triangular.' It's best to illustrate it just with an example.

$$\begin{aligned} x - y + z &= 0 \\ 2y - z &= 1 \\ 3z &= -1 \end{aligned}$$

  We observe that $z = -1/3$, and then substitute that into the next equation from the bottom to get that $2y - (-1/3) = 1$, or that $y = 1/3$. We can then substitute both of those quantities into the top equation to get $x - (1/3) + (-1/3) = 0$, so $x = 0$. This gives us a solution $\mathbf{v} = [0, 1/3, -1/3]$.

- A $m \times n$ **matrix** is a grid of m rows and n columns. Each position in the matrix is filled by a real number, which we call an **entry** of the matrix.

  Example The following is a 2×3 matrix:

$$\begin{pmatrix} 1 & 4 & 3 \\ 1 & 1 & 1 \end{pmatrix}$$

- A **coefficient matrix** corresponding to a system of m equations with n variables is a $m \times n$ matrix where with entry in the ith row and jth column is the coefficient of the jth variable in the ith equation.

  Example the coefficient matrix corresponding to this system:

$$\begin{aligned} x - y + z &= 0 \\ 2y - z &= 1 \\ 3z &= -1 \end{aligned}$$

  is:

$$\begin{pmatrix} 1 & -1 & 1 \\ 0 & 2 & -1 \\ 0 & 0 & 3 \end{pmatrix}$$

- An **augmented matrix** corresponding to a system of m equations with n variables is a $m \times n + 1$ matrix where the left n columns consists of the coefficient matrix and the rightmost column is the constants of each of the equations.

  Example The augmented matrix corresponding to the above system is:

$$\left(\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 2 & -1 & 1 \\ 0 & 0 & 3 & -1 \end{array}\right)$$