Homework 7 Solutions

5.1.36 If $n > m$ then there is no $m \times n$ matrix A such that $\|A\mathbf{x}\| = \|\mathbf{x}\|$. (Hint: this has not much to do with norms).

Solution. By the rank-nullity theorem, the nullity must be larger than 0. $\mathbf{x} \in \text{null}(A)$ where $\mathbf{x} \neq \mathbf{0}$. Then $\|\mathbf{x}\| \neq 0$ but $\|A\mathbf{x}\| = \|\mathbf{0}\| = 0$.

2 Prove that if $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ is an orthogonal set and c is a nonzero scalar then $S' = \{\mathbf{v_1}, \ldots, c\mathbf{v_i}, \ldots, \mathbf{v_k}\}$ is a set of orthogonal set.

Proof. If $v_n, v_m \in S'$ then: if $\mathbf{v}_n \neq \mathbf{v}_i$ and $\mathbf{v}_m \neq \mathbf{v}_i$ then we know $\mathbf{v}_n \cdot \mathbf{v}_m = 0$ as they were in S.

If f $\mathbf{v}_n = \mathbf{v}_i$ then we have $\mathbf{v}_n \cdot \mathbf{v}_m = c\mathbf{v}_i \cdot \mathbf{v}_m = c(\mathbf{v}_i \cdot \mathbf{v}_m) = 0$ we they are both in S.

5.1.11 and 5.1.12 Determine whether the given vectors are orthonormal. If they are not, then normalize them to form an orthonormal set.

1. $\begin{pmatrix} 3/5 \\ 4/5 \end{pmatrix}, \begin{pmatrix} -4/5 \\ 3/5 \end{pmatrix}$
2. $\begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix}, \begin{pmatrix} 1/2 \\ -1/2 \end{pmatrix}$

Solution.

1. They are orthonormal:

$$\left\| \begin{pmatrix} 3/5 \\ 4/5 \end{pmatrix} \right\| = \sqrt{(3/5)^2 + (4/5)^2} = 1$$

$$\left\| \begin{pmatrix} -4/5 \\ 3/5 \end{pmatrix} \right\| = \sqrt{(-4/5)^2 + (3/5)^2} = 1$$

$$\begin{pmatrix} 3/5 \\ 4/5 \end{pmatrix} \cdot \begin{pmatrix} -4/5 \\ 3/5 \end{pmatrix} = 0$$

2. They are not orthogonal:

$$\left\| \begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix} \right\| = \sqrt{(1/2)^2 + (1/2)^2} = 1/\sqrt{2}$$

$$\left\| \begin{pmatrix} 1/2 \\ -1/2 \end{pmatrix} \right\| = \sqrt{(1/2)^2 + (-1/2)^2} = 1/\sqrt{2}$$

We normalize them:

$$\frac{1}{\left\| \begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix} \right\|} \begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix} = \begin{pmatrix} \sqrt{2}/2 \\ \sqrt{2}/2 \end{pmatrix}$$

$$\frac{1}{\left\| \begin{pmatrix} 1/2 \\ -1/2 \end{pmatrix} \right\|} \begin{pmatrix} 1/2 \\ -1/2 \end{pmatrix} = \begin{pmatrix} \sqrt{2}/2 \\ -\sqrt{2}/2 \end{pmatrix}$$

5.1.17 Determine whether the given matrix is orthogonal. If it is, then find it's inverse.

$$\begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}$$

Solution. By the previous problem, the columns are orthonormal. Or, another way, you can verify that the transpose is in inverse:

$$A^T = \begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}$$

5.1.26 If Q is an orthogonal matrix, prove that any matrix obtained by rearranging the rows of Q is also orthogonal.

Proof. Let Q' be Q with it's rows rearranged. Let q_i' be the ith column of Q', and q_i be the ith column of q. Then:

$$\|q_i'\|^2 = \sum_{j=1}^{n}(q_{ij}')^2 = \sum_{j=1}^{n}(q_{ij} = \|q_i\|^2 = 1$$

Where the equality between the summation occurs since one sum is merely a permutation of the other. So the columns are still unit vectors.

We next claim they are still othogogonal. Let q_k and q_k' be the kth column of Q and Q' respectively. Then:

$$q_i' \cdot q_k' = \sum_{j=1}^{n} q_{ij}' q_{kj}' = \sum_{j=1}^{n} q_{ij} q_{kj} = q_i \cdot q_k$$

which is 0 if $i = k$ and 0 otherwise, as Q was orthogonal. □

5.2.14 Let W be the subspace spanned by:

$$\mathbf{w}_1 = \begin{pmatrix} 3 \\ 2 \\ 0 \\ -1 \\ 4 \end{pmatrix} \qquad \mathbf{w}_2 = \begin{pmatrix} 1 \\ 2 \\ -2 \\ 0 \\ 1 \end{pmatrix} \qquad \mathbf{w}_3 = \begin{pmatrix} 3 \\ -2 \\ 6 \\ -2 \\ 5 \end{pmatrix}$$

Find a basis for $W^\perp$

Solution. We put the vectors as row vectors as find the null space since $(\text{row}(A))^\perp = (A)$.

$$\begin{pmatrix} 3 & 2 & 0 & -1 & 4 \\ 1 & 2 & -2 & 0 & 1 \\ 3 & -2 & 6 & -2 & 5 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 1 & -1/2 & 3/2 \\ 0 & 1 & -3/2 & 1/4 & -1/4 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

So a basis for the null space is: $\left\{ \begin{pmatrix} -1 \\ 3/2 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1/2 \\ -1/4 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -3/2 \\ 1/4 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}$

5.3.6 Use Gram-Schmidt Process to find an orthogonal basis for W where W is the span of:

$$\mathbf{x}_1 = \begin{pmatrix} 1 \\ 2 \\ -2 \\ 1 \end{pmatrix} \qquad \mathbf{x}_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 2 \end{pmatrix} \qquad \mathbf{x}_3 = \begin{pmatrix} 1 \\ 8 \\ 1 \\ 0 \end{pmatrix}$$

Solution. We will find an orthonormal basis because, well, why not.

$$\|\mathbf{x}_1\| = \sqrt{10}$$

So we set:

$$\mathbf{v}_1 = \frac{1}{\sqrt{10}} \begin{pmatrix} 1 \\ 2 \\ -2 \\ 1 \end{pmatrix}$$

Then

$$\mathbf{v}_2 = \mathbf{x}_2 - \text{proj}_{x_1}(x_2)$$

So we calculate:

$$\text{proj}_{\mathbf{x}_1}(\mathbf{x}_2) = (\mathbf{x}_2 \cdot \mathbf{x}_1')\mathbf{x}_1' = \begin{pmatrix} 1/2 \\ 1 \\ -1 \\ 1/2 \end{pmatrix}$$

so, we get:

$$\mathbf{x}_2 - \text{proj}_(\mathbf{x}_2) = \begin{pmatrix} 1/2 \\ 0 \\ 1 \\ 3/2 \end{pmatrix}$$

And we can normalize it (cause why not) and get:

$$\mathbf{v}_2 = 2/\sqrt{14} \begin{pmatrix} 1/2 \\ 0 \\ 1 \\ 3/2 \end{pmatrix}$$

Then we can do the same for x_3 and get:

$$\mathbf{x}_3 - \text{proj}_{\mathbf{v}_1}(\mathbf{x}_3) - \text{proj}_{\mathbf{v}_2}(\mathbf{x}_3) = \begin{pmatrix} -5/7 \\ 5 \\ 25/7 \\ -15/7 \end{pmatrix}$$

And we can normalize it:

$$\mathbf{v}_3 = \sqrt{7/300} \begin{pmatrix} -5/7 \\ 5 \\ 25/7 \\ -15/7 \end{pmatrix}$$

So an orthogonal basis is:

$$\left\{ \frac{1}{\sqrt{10}} \begin{pmatrix} 1 \\ 2 \\ -2 \\ 1 \end{pmatrix}, 2/\sqrt{14} \begin{pmatrix} 1/2 \\ 0 \\ 1 \\ 3/2 \end{pmatrix}, \sqrt{7/300} \begin{pmatrix} -5/7 \\ 5 \\ 25/7 \\ -15/7 \end{pmatrix} \right\}$$

5.3.10 Use Gram-Schmidt Process to find an orthogonal basis for the column space of:

$$A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

Solution. We set:

$$\mathbf{v}_1 = \mathbf{x}_1 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

Then we let

$$\mathbf{v}_2 = \mathbf{x}_2 - \text{proj}_{\mathbf{v}_1}(\mathbf{x}_2) = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} - \begin{pmatrix} 0 \\ 1/2 \\ 1/2 \end{pmatrix} = \begin{pmatrix} 1 \\ -1/2 \\ 1/2 \end{pmatrix}$$

Then we set

$$\mathbf{v}_3 = \mathbf{x}_3 - \text{proj}_{\mathbf{v}_1}(\mathbf{x}_3) - \text{proj}_{\mathbf{v}_2}(\mathbf{x}_3) = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ 1/2 \\ 1/2 \end{pmatrix} - \begin{pmatrix} 1/3 \\ -1/6 \\ 1/6 \end{pmatrix} = \begin{pmatrix} 2/3 \\ 2/3 \\ -2/3 \end{pmatrix}$$

So an orthogonal basis is

$$\left\{ \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1/2 \\ 1/2 \end{pmatrix}, \begin{pmatrix} 2/3 \\ 2/3 \\ -2/3 \end{pmatrix} \right\}$$

5.3.12 Find an orthogonal basis for $\mathbb{R}^4$ that contains:

$$\begin{pmatrix} 2 \\ 1 \\ 0 \\ -2 \end{pmatrix} \text{ and } \begin{pmatrix} 1 \\ 0 \\ 3 \\ 2 \end{pmatrix}$$

Solution. So we will take these two vectors and find a basis for the remainder of the space. This is the perp. So first we find a basis for the span of these two vectors:

$$\begin{pmatrix} 2 & 1 & 0 & -2 \\ 1 & 0 & 3 & 2 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 3 & 2 \\ 0 & 1 & -6 & -6 \end{pmatrix}$$

A basis for the null space is:

$$\left\{ \begin{pmatrix} -3 \\ 6 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ 6 \\ 0 \\ 1 \end{pmatrix} \right\}$$

We now want to find an orthogonal basis for this subspace using Gram-Schmidt. We take:

$$\mathbf{v}_3 = \begin{pmatrix} -3 \\ 6 \\ 1 \\ 0 \end{pmatrix}$$

Then we do:

$$\begin{pmatrix} -2 \\ 6 \\ 0 \\ 1 \end{pmatrix} - \text{proj}_{\mathbf{v}_3} \begin{pmatrix} -2 \\ 6 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 17/23 \\ 12/23 \\ -21/23 \\ 1 \end{pmatrix}$$

We can scale that by 23 if we choose. Then we end up with the basis:

$$\left\{ \begin{pmatrix} 2 \\ 1 \\ 0 \\ -2 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 3 \\ 2 \end{pmatrix}, \begin{pmatrix} -3 \\ 6 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 17 \\ 12 \\ -21 \\ 23 \end{pmatrix} \right\}$$