Homework 6 Solutions

1 Calculate $\det(A)$ if:

$$A = \begin{pmatrix} 1 & 1 & 3 & 11 \\ 1 & 3 & 7 & 21 \\ -2 & 0 & 0 & -4 \\ 0 & -2 & -2 & -6 \end{pmatrix}$$

Solution. Do some row reductions (which all involve adding a row to a constant multiple of another row), and you arrive at the matrix:

$$\begin{pmatrix} 1 & 1 & 3 & 11 \\ 0 & 2 & 4 & 10 \\ 0 & 0 & 2 & 8 \\ 0 & 0 & 0 & -4 \end{pmatrix}$$

As this matrix was obtained from row reductions that didn't involve scaling or switching rows, they have the day determinant. As this matrix is upper triangular, it's determinant is the product along the diagonal:

$$1 \cdot 2 \cdot 2 \cdot -4 = -16$$

2 A block of questions about finding the determinant by inspection:

4.2.26

$$\det \begin{pmatrix} 1 & 1 & 1 \\ 3 & 0 & -2 \\ 2 & 2 & 2 \end{pmatrix}$$

Solution. Half of row three is identitical to row one. So the determinant is 0.

4.2.28

$$\det \begin{pmatrix} 0 & 0 & 1 \\ 0 & 5 & 2 \\ 3 & -2 & 4 \end{pmatrix}$$

Solution. Switch row one and three. Then the matrix is upper triangular. The act of switching two rows cost a negative, so the determiant is $-(3)(5)(1) = -15$

4.2.30

$$\det \begin{pmatrix} 1 & 2 & 3 \\ 0 & 4 & 1 \\ 1 & 6 & 4 \end{pmatrix}$$

Solution. Row three minus row one minus row two has a 0 row. As this doesn't change the determinant, the determinant is 0.

4.2.32

$$\det \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Solution. Switch row 2 and three. This is upper triangular. Switching two rows switches sign. So determinant is -1

4.2.34

$$\det \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

Solution. Add column 4 to column 1 and column 3 to column 2. This doesn't change the determinant, but there are two identical columns. Thus the determinant is 0.

4.2.56 Find $\det(A)$ if A is nilpotent (i.e. $A^m = O$ for some $m > 1$).

Proof. It is 0. $\det(A^m) = \det(O) = 0$. But, $\det(A^m) = \det(A)^m$. So $\det(A)^m = 0$, so $\det(A) = 0$. $\square$

4.3.2 Find (a) characteristic polynomial, (b) spectrum, (c) bases for the eigenspaces, and (d) geometric and algebraic multipicites for each eigenvalue.

$$\begin{pmatrix} 1 & -9 \\ 1 & -5 \end{pmatrix}$$

Solution.

$$\det \begin{pmatrix} 1-\lambda & -9 \\ 1 & -5-\lambda \end{pmatrix} = (\lambda+2)^2$$

This is the characteristic polynomial. The spectrum is $\{-2\}$. This eigenvalue has algebraic multiplicity 2. To find it's eigenspace, we find the nullspace of:

$$\begin{pmatrix} 3 & -9 \\ 1 & -3 \end{pmatrix}$$

This is $t\begin{pmatrix} 3 \\ 1 \end{pmatrix}$. So a basis for the eigenspace is $\left\{\begin{pmatrix} 3 \\ 1 \end{pmatrix}\right\}$. This is geometric multiplicity 1.

4.3.4 Find (a) characteristic polynomial, (b) spectrum, (c) bases for the eigenspaces, and (d) geometric and algebraic multipicites for each eigenvalue.

$$\begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

Solution.

$$\det \begin{pmatrix} 1-\lambda & 1 & 0 \\ 1 & -\lambda & 1 \\ 0 & 1 & 1-\lambda \end{pmatrix} = -(\lambda-1)(\lambda+1)(\lambda-2)$$

This is the characteristic polynomial. The spectrum is $\{1, -1, 2\}$. Each has algebraic multiplicity 2.

We find the eigenspace of $\lambda = 1$ by finding nullspace of:

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

This has nullspace $t\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$ which has basis $\left\{\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}\right\}$. So $\lambda = 1$ has geometric multiplicity 1.

We find the eigenspace of $\lambda = -1$ by finding nullspace of:

$$\begin{pmatrix} 2 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 2 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}$$

This has nullspace $t\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$ which has basis $\left\{\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}\right\}$. So $\lambda = -1$ has geometric multiplicity 1.

We find the eigenspace of $\lambda = 2$ by finding nullspace of:

$$\begin{pmatrix} -1 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}$$

This has nullspace $t\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ which has basis $\left\{\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}\right\}$. So $\lambda = -1$ has geometric multiplicity 1.

4.3.22 Let $\mathbf{v}$ be an eigenvector of A with λ has an eigenvalue. Let c be a constant. Prove that $\mathbf{v}$ is an eigenvector of $A - cI$ with eigenvalue $\lambda - c$.

Proof.

$$\begin{aligned}(A - cI)\mathbf{v} &= A\mathbf{v} - c\mathbf{v} \\ &= \lambda v - c\mathbf{v} \\ &= (\lambda - c)\mathbf{v}\end{aligned}$$

□

4.3.10 Determine if diagonalizable. If it is, find it:

$$\begin{pmatrix}1 & 2 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 2\end{pmatrix}$$

Solution. By inspection, the eigenvalues are $1, 2$, and 1 has algebraic multiplicity 2. So, we know if 1 had geometric multiplicity 2 it would be diagonalizable. So we look at the eigenspace of 1 by finding the nullspace of:

$$\begin{pmatrix}0 & 2 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 2\end{pmatrix}$$

This has nullspace $t\begin{pmatrix}1 \\ 0 \\ 0\end{pmatrix}$. In particular, it has geometric multiplicity 1, and so it is not diagonalizable.

4.4.42 Let A be diagonalizable where each eigenvalue is 0 or 1. Prove A is idempotent.

Proof. First, note that if D is any diagonal matrix of with only 0s and 1s along the diagonal it is idempotent. This is because If A and B are diagonal then AB is a matrix with the (i,i) coordinate is $a_{ii}b_{ii}$, which in out case is a_{ii}^2, and moreover since a_{ii} is either 0 or 1 we have $a_{ii}^2 = a_{ii}$. So, $D^2 = D$ for any diagonal matrix D with 0s and 1s along the diagonal.

A is diagonalizable. So there is a D and P such that $A = PDP^{-1}$. So $A^2 = PD^2P^{-1}$. Moreover, D is just the eigenvalues of A along the diagonal. So D is a diagonal matrix with 0s and 1s along diagonal, we have:

$$A^2 = PD^2P^{-1} = PDP^{-1} = A$$

So A is idempotent. □