Homework 5 Solutions

3.6.7 Give a counterexample to show that the given transformation is not a linear transformation:

$$T\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}y\\x^2\end{pmatrix}$$

Solution. Note:

$$T\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}0\\1\end{pmatrix}$$
$$T\begin{pmatrix}0\\2\end{pmatrix} = \begin{pmatrix}0\\4\end{pmatrix}$$

So:

$$T\begin{pmatrix}0\\1\end{pmatrix} + T\begin{pmatrix}0\\2\end{pmatrix} = \begin{pmatrix}0\\5\end{pmatrix}$$

But

$$T\left(\begin{pmatrix}0\\1\end{pmatrix} + \begin{pmatrix}0\\2\end{pmatrix}\right) = T\begin{pmatrix}0\\3\end{pmatrix} = \begin{pmatrix}0\\9\end{pmatrix}$$

3.6.44 Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be a linear transformation. Show that T maps straight lines to a straight line or a point.

Proof. In $\mathbb{R}^3$ we can represent a line as:

$$\mathbf{x} = t\mathbf{m} + \mathbf{b}$$

Where $\mathbf{m} \neq \mathbf{0}$. So,

$$T(t\mathbf{m} + \mathbf{b}) = t(T\mathbf{m}) + T(\mathbf{b})$$

If $T\mathbf{m} = 0$ (i.e. $\mathbf{m} \in \ker(T)$) then T sends the line to a point, namely $T\mathbf{b}$. Otherwise, $T\mathbf{m} = \mathbf{k} \neq \mathbf{0}$ and $T\mathbf{b} = \mathbf{c}$ So we have the line gets sent to

$$t\mathbf{k} + \mathbf{c}$$

which is a line in $\mathbb{R}^3$. □

3.6.53 Prove that $T : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation if and only if

$$(*) \qquad T(c_1\mathbf{v_1} + c_2\mathbf{v_2}) = c_1T(\mathbf{v_1}) + c_2(\mathbf{v_2})$$

for all vectors $\mathbf{v_1}, \mathbf{v_2} \in \mathbb{R}^n$ and scalars c_1, c_2.

Proof. ($\Leftarrow$) We need to show that T respects scalar multiplication and scalar multiplication.

- First we show that for any $\mathbf{x}, \mathbf{y}$ we have $T(\mathbf{x} + \mathbf{y}) = T\mathbf{x} + T\mathbf{y}$. From the property $(*)$ where $c_1 = c_2 = 1$ and $\mathbf{v_1} = \mathbf{x}$ and $\mathbf{v_2} = \mathbf{y}$ we have that

$$T(1\mathbf{x} + 1\mathbf{y}) = 1T\mathbf{x} + 1T\mathbf{y} = T\mathbf{x} + T\mathbf{y}$$

- Need we show that for any $\mathbf{x}$ and scalar c we have $T(c\mathbf{x}) = cT\mathbf{x}$. We use $(*)$ for $c_1 = c$, $c_2 = 0$, $\mathbf{v_1} = \mathbf{v_2} = \mathbf{x}$ and we get:

$$T(c\mathbf{x} + 0\mathbf{x}) = cT\mathbf{x} + 0T\mathbf{x} = cT\mathbf{x}$$

($\Rightarrow$) Let $\mathbf{v_1}, \mathbf{v_2} \in \mathbb{R}^n$ and c_1, c_2 be scalars. Then we want to show

$$T(c_1\mathbf{v_1} + c_2\mathbf{v_2}) = c_1T(\mathbf{v_1}) + c_2(\mathbf{v_2})$$

Well, $T(c_1\mathbf{v_1} + c_2\mathbf{v_2}) = T(c_1\mathbf{v_1}) + T(c_2\mathbf{v_2})$ by the sum property of linearity. Then $T(c_1\mathbf{v_1}) + T(c_2\mathbf{v_2}) = c_1T\mathbf{v_1} + c_2T\mathbf{v_2}$ by the scalar proprety. This is what we wanted. □

4.1.12 Show that $\lambda = 3$ is an eigenvalue for the following matrix, and find one eigenvector.

$$A = \begin{pmatrix} 3 & 1 & -1 \\ 0 & 1 & 2 \\ 4 & 2 & 0 \end{pmatrix}$$

Solution. Consider $A - 3I$:

$$A - 3I = \begin{pmatrix} 0 & 1 & -1 \\ 0 & -2 & 2 \\ 4 & 2 & -3 \end{pmatrix}$$

Doing the elementary row operation of adding -2 times row 1 to row 2 we get:

$$\begin{pmatrix} 0 & 1 & -1 \\ 0 & -2 & 2 \\ 4 & 2 & -3 \end{pmatrix} \to \begin{pmatrix} 0 & 1 & -1 \\ 0 & 0 & 0 \\ 4 & 2 & -3 \end{pmatrix}$$

This show that the matrix does not have full rank, so therefore has a nontrivial null space. This is enough to know that $\lambda = 3$ is an eigenvalue. Continuing row reductions to rref:

$$\begin{pmatrix} 1 & 0 & -1/4 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}$$

We now just need to find some vector in the null space. Viewing this has a homogenious equation, we see that the solutions look like:

$$t \begin{pmatrix} 1/4 \\ 1 \\ 1 \end{pmatrix}$$

These are all the eigenvectors. One particular eigenvector is $\begin{pmatrix} 1 \\ 4 \\ 4 \end{pmatrix}$ 4

4.1.24 Use determinants of 2×2 matrices to find the spectrum of the given matrix. Find the eigenspaces and then give bases for each eigenspace.

$$A = \begin{pmatrix} 0 & 2 \\ 8 & 6 \end{pmatrix}$$

Solution. Consider the following matrix:

$$A - \lambda I = \begin{pmatrix} -\lambda & 2 \\ 8 & 6 - \lambda \end{pmatrix}$$

Calculating the determinant:

$$\det(A - \lambda I) = -\lambda(6 - \lambda) - 16 = \lambda^2 - 6\lambda - 16 = (\lambda - 8)(\lambda + 2)$$

The matrix is invertible if and only if the determinant is nonzero. Thus, this has a nontrivial null space only when the determinant is zero. Thus is has eigenvalues when the determinant is zero: $\lambda_1 = 8$, $\lambda_2 = -2$.

We can then find eigenvectors by calculating the actual nullspace of $A - 8I$ and $A + 2I$. The former is the matrix:

$$\begin{pmatrix} -8 & 2 \\ 8 & -2 \end{pmatrix}$$

The nullspace of this matrix is:

$$t \begin{pmatrix} 1/4 \\ 1 \end{pmatrix}$$

This is just a one dimensional space, so a basis for the eigenspace corresponding to 8 is just $\begin{pmatrix} 1/4 \\ 1 \end{pmatrix}$ (or any nonzero multiple of this vector will do) And for the other, we get the matrix:

$$\begin{pmatrix} 2 & 2 \\ 8 & 8 \end{pmatrix}$$

The nullspace of this matrix is:

$$t\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

This is just a one dimensional space, so a basis for the eigenspace corresponding to -2 is just $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ (or any nonzero multiple of this vector will do)

4.1.37 Show that the eigenvalues of the upper triangular matrix:

$$A = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$$

are $\lambda = a$ and $\lambda = d$. Find the corresponding eigenspaces.

Solution. Consider $A - aI$.

$$A - aI = \begin{pmatrix} 0 & b \\ 0 & d-a \end{pmatrix}$$

This matrix has a nontrivial null space since it's rank is less than the number of columns. Moreover, assuming that $b \neq 0$ or $d - a \neq 0$, then the rank of the matrix is 1, so it's nullspace is one dimensional given by:

$$t\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

In the event both are zero, the rank is 0, and it's nullspace is 2 dimensional and is given by:

$$t\begin{pmatrix} 1 \\ 0 \end{pmatrix} + s\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Similarly, $A - dI$ is:

$$A - dI = \begin{pmatrix} a-d & b \\ 0 & 0 \end{pmatrix}$$

if $a - d \neq 0$ or $b \neq 0$ we have that the rank of the matrix is 1, so it has a one dimensional nullspace. If $a - d \neq 0$ then it's nullspace is given by:

$$t\begin{pmatrix} -\frac{b}{a-d} \\ 1 \end{pmatrix}$$

Otherwise, it's simply

$$t\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

In the even that both are 0 then the rank of the matrix is 0 in which case the null space is two dimensional, so it is:

$$t\begin{pmatrix} 1 \\ 0 \end{pmatrix} + s\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$