Homework 3 Solutions

3.5.2 Determine is the set of all $\{(x, y) \in \mathbb{R}^2 \mid x \geq 0 \text{ and } y \geq 0\}$ is a subspace of $\mathbb{R}^2$

Solution. Not a subspace. $\begin{pmatrix} 1 \\ 1 \end{pmatrix} \in S$ but $-\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \end{pmatrix}$. Therefore S is not closed under scalar multiplication.

3.5.6 Determine if $\begin{pmatrix} x \\ y \\ z \end{pmatrix} \subseteq \mathbb{R}^3$ such that $z = 2x$ and $y = 0$ form a subspace of $\mathbb{R}^3$.

Solution. S is a subspace.

1. $\mathbf{0} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ so $x = 0$ and $z = 0$. Since $y = 0$ and $z = 2x$, we have $0 \in S$.
2. Let $\mathbf{v_1}, \mathbf{v_2} \in S$. Then they can be written as

$$\mathbf{v_1} = \begin{pmatrix} x_1 \\ 0 \\ 2x_1 \end{pmatrix} \qquad \mathbf{v_2} = \begin{pmatrix} x_2 \\ 0 \\ 2x_2 \end{pmatrix}$$

   So:

$$\mathbf{v_1} + \mathbf{v_2} = \begin{pmatrix} x_1 + x_2 \\ 0 \\ 2(x_1 + x_2) \end{pmatrix} \in S$$

3. Let $\mathbf{v_1} \in S$; so $\mathbf{v_1} = \begin{pmatrix} x \\ 0 \\ 2x \end{pmatrix}$. Let c be a scalar. So $c\mathbf{v_1} = \begin{pmatrix} cx \\ 0 \\ 2cx \end{pmatrix} \in S$

3.5.40 If A is a 4×2 matrix, explain why the rows of A must be linearly dependent.

Solution. The rows of A are vectors in $\mathbb{R}^2$. If you have more then 2 vectors of $\mathbb{R}^2$ then they are linearly dependent.

3.5.58 If A and B have rank n then AB has rank n.

Proof. There are many ways you can prove this. Here is one way:

Let $C = AB$. Let $v \in \mathbb{R}^n$ so that $C\mathbf{v} = 0$. We want to show that $\mathbf{v} = \mathbf{0}$. Well, $(AB)\mathbf{v} = 0$. So, by associativity $A(B\mathbf{v}) =$. As A has rank n, $B\mathbf{v} = \mathbf{0}$. As B has rank n, $\mathbf{v} = \mathbf{0}$. □

3.5.28 Find a basis for the span of these vectors:

$$\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} \right\}$$

Solution. We will put them as row vectors in a matrix and row reduce

$$\begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 2 & 2 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

There are many reductions there. Therefore, a basis is:

$$S = \{[1, 0, 0], [0, 1, 0], [0, 0, 1]\}$$

(Remark: these vectors span the entire space $\mathbb{R}^3$)

3.5.35 Calculate the rank and the nullity:

$$\begin{pmatrix} 1 & 0 & -1 \\ 1 & 1 & 1 \end{pmatrix}$$

Solution. We first determine a basis of the column space by putting the vectors as the rows of a matrix and doing row reduction:

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \\ -1 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$$

So we have that a basis for the column space is $\{[1,0],[0,1]\}$ so its dimension is 2.

By the rank nullity theorem, the null space is dimension 1. But, if you want to do it out another way, consider the matrix:

$$\begin{pmatrix} 1 & 0 & -1 \\ 1 & 1 & 1 \end{pmatrix}$$

Row reduce it to find solutions to the homogeneous equation it represents.

$$\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \end{pmatrix}$$

Solving, one sees that the solutions are all:

$$\mathbf{x} = t\begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

Therefore, the vector $[1,-2]$ spans the null space, so it's dimension is 1

3.5.52 Show that $\begin{pmatrix} 1 \\ 3 \\ 4 \end{pmatrix}$ can be written as a linear combination of $\left\{ \begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix}, \begin{pmatrix} 5 \\ 1 \\ 6 \end{pmatrix} \right\}$ and write the coordinate vector with respect to this basis.

Solution. We will show this by setting up the following augmented matrix:

$$\left(\begin{array}{cc|c} 3 & 5 & 1 \\ 1 & 1 & 3 \\ 4 & 6 & 4 \end{array}\right)$$

Doing row reductions, we get:

$$\left(\begin{array}{cc|c} 1 & 0 & 7 \\ 0 & 1 & -4 \\ 0 & 0 & 0 \end{array}\right)$$

This tells us that

$$7\begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix} - 4\begin{pmatrix} 5 \\ 1 \\ 6 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ 4 \end{pmatrix}$$

So the coordinate vector with respect to this basis is $\begin{pmatrix} 7 \\ -4 \end{pmatrix}$