Day 2

Tuesday May 22, 2012

Objectives:

- Review properties of numbers.
- Practice basic proofs
- Learn about some particularly important inequalities.

1 Numbers

Yesterday, we ended class talking about the real numbers. Let's begin this class with some examples of real numbers, as well as some properties.

1.1 Real Numbers

We already had one example of a number that is real that is not rational.

Theorem 1. *The length of the diagonal of a 1 by 1 square is real but not rational.*

This is a theorem that we will prove in a few days after we have a good understanding of proofs. For now however, we take that as our first example of a number which is not rational, or **irrational**.

There are a few more examples, such as:

- π, which is the ratio of the circumference of a circle to it's diameter.
- e, which is the base of the natural logarithm.
- $\ln(2)$, which is the power one must raise the natural exponential base to in order to get 2.
- $\sqrt[3]{2}$, which is the side length requires to make a cube of volume 2.

And many more.

2 Inequalities

2.1 Function Review

First, we will review what a real-valued function is. Functions are the a major topic of this course, and mathematics in general. Today, we just want to raise intuition, which hopefully we already have some of.

Definition 1. A **real-valued function** is an assignment, which assigns every real value on its domain to exactly one real value. These are normally (although do not need to be given) by a rule, or explicit assignment.

Example 1. An example of a real-valued function is $f(x)$ given by the rule

$$f(x) := x^2$$

This function takes in a real value, and then squares it.

A non-example of a real-value function is

$$f(x)^2 := x$$

This does yield an assignment that can be read off (implicitly). But, you can see that on input 1 there are two possible outputs that could satisfy the equation: -1 and 1. This violates the "exactly one real value" condition of the definition.

Let's take a minute, and remember some of the function that you might be acquainted with.

- Polynomials: $f(x) := a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$
- Trig function: $\sin(x), \cos(x), \tan(x)$.
- Exponential functions: $f(x) := a^x$

2.2 Absolute Value

We begin with a look at a function which we haven't mentioned yet called the absolute value function.

Definition 2. For a real number x, we defined the real valued function $|x|$, which we call the **absolute value function** by

$$|x| := \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{otherwise} \end{cases}$$

Question 1. What do you think is the intended purpose of the absolute value function?

Answer 1. The absolute value function is intended to represent the distance from a number to the number 0 on the number line!

Let us begin by proving some properties of this function.

Theorem 2. *For every real number x, $|x| \geq 0$.*

This theorem tells us that the absolute value is always at least as large as 0. It is worth talking about this exactly.

Question 2. If this theorem were true, does that mean that $|x|$ has a minimum at 0?

Answer 2. No, not by itself. This theorem tells us that $|x|$ is **bounded from below** by 0. In order for 0 to be a minimum, it must be attained, ie. there must be a value that maps to it. This is the case though, so 0 is indeed a minimum of 0.

Okay, now. How do you propose we prove this theorem?

Proof. As the absolute function is defined by cases, it is very natural to prove the theorem by cases on x.
Case 1: $x \geq 0$.
In this case, $x \geq 0$, and $|x| = x$, therefore $|x| \geq 0$.
Case 2: $x < 0$.
In this case, $|x| = -x$. As x is negative, $-x$ is positive, and thus, $-x > 0$. □

We now know that $|x|$ is always positive. x, however, need not be positive. So, we might ask: How does $|x|$ compare with x?

Theorem 3. $|x| \geq x$ *for all* x.

Proof. Here, we do cases, as above.
Case 1: $x \geq 0$.
In this case, $|x| = x$, and therefore $|x| \geq x$.
Case 2: $x < 0$.
In this case, $|x| = -x$, and as x is negative, $-x$ is positive. Positive numbers are always larger than negative numbers, and thus $|x| = -x > x$. □

Question 3. As with many proofs, we can "read off" more from the proof than we wrote in the theorem. What do we know that's even stronger than we proved?

Answer 3. We know that $|x| = x$ when x is non-negative (ie. positive or 0), and the inequalities we proved is **strict** (ie. $>$ as opposed to $\geq$) when x is negative.

Now, we prove a theorem which is very important in many branches of mathematics. This theorem is called the **triangle inequality**. What it says is that if one takes two real numbers, and looks as them as points on the number line, then the distance it would take you to start at 0 and go to a then to start at 0 and go to b is at least as much distance from 0 to $a + b$.

Theorem 4 (Triangle Inequality)**.** *For every x, y real numbers, we have*

$$|x + y| \leq |x| + |y|$$

Remark 1. Notice, what I said the theorem said is not at all like it's written about. It's important to be able to think about the meaning of a theorem and be able to translate that in a more abstract statement. Likewise, one must also be comfortable with looking at the abstract stuff and translating it into everyday language. We will gain practice on this throughout the semester!

What are some of your ideas on how to prove the triangle inequality?

Proof 1. One idea may be to enumerate all the cases on how a and b are comparable to 0. Cases are nice, because they allow us extra assumptions and also added insight (eg. what values of a and b make it nontrivial?). The downside of cases are that they can make the proof very long. Also, if one is sloppy with how one displays information, it make a proof kind of unmanageable.

In general though, thinking about the easy cases is a good thing.

So, let's do cases.

Case 1: $a, b \geq 0$

This is the most straightforward case, because if both are positive, then $|a + b| = a + b = |a| + |b|$ and we are done.

Case 2: $a, b \leq 0$

This case is equally easy, because what is written above is still essentially true.

$$|a + b| = -a - b = |a| + |b|$$

Case 3: $a \leq 0$ and $b \geq 0$.

Now, don't know what exactly $a + b$ is. Imagine however we are b. We know when we get a added to us, we get smaller, so we know $b > a + b$. We want to conclude that $|b| > |a + b|$, but we cannot because a might be very negative, and might have made this number very small. In a way, we see that our current path is stuck unless we know more about $a + b$. Therefore, we do more cases.

Case 3.1: $a + b > 0$

Now we have even more information. We have already seen $b > a + b$, and now we know that $a + b$ is itself positive. Therefore, $|b| > |a + b|$. Here, we are already done though, because adding something positive a quantitity which is already larger than another quantity preserves that inequality; that is, since $|a| > 0$:

$$|a + b| < |b| < |b| + |a|$$

Case 3.2 $a + b < 0$

The roles of a and b switch in this case. Now, making the same observations in the beginning of case 3 we see that $a < a + b$. But of course, when you take the absolute value a smaller negative becomes a larger positive, so $|a| > |a + b|$. Then we have

$$|a + b| < |a| < |b| + |a|$$

Case 3.3 $a + b = 0$

This is an easy case to forget, but without it the proof would not be complete. If $a + b = 0$, then $|a + b| = 0$. $0 \leq p$ for any positive number p, so as $|a| + |b| > 0$

$$|a + b| = 0 \leq |a| + |b|$$

Case 4: $a > 0$ and $b < 0$

Notice, we will be doing the same work here as we did in case 3.1 and 3.2. As an exercise, you should do this case.

□

Proof. We can shorten the above and really think about what cases we need to get the job done.

Case 1: $a + b \geq 0$

If $a + b \geq 0$ then $|a + b| = a + b$. By the theorem above, we have $a \leq |a|$ and $b \leq |b|$. But then we have $a + b \leq |a| + |b|$. Chaining these together, we get

$$|a + b| = a + b \leq |a| + |b|$$

Case 2: $a + b < 0$

Here, we nice that $-a - b > 0$, and moreover $|a + b| = | - a - b|$. Therefore, we have that

$$|a + b| = | - a - b| = -a - b \leq | - a| + | - b| = |a| + |b|$$

□

2.3 Squares

We've already talked about the squaring function. Now, let's state an arithmetical fact that requires no proof:

Fact 1. $x^2 \geq 0$ for every real number x.

There is no intuitive reason for this fact. Geometrically, the squaring operation gives a area, but since negative sides don't make sense, there's no reason to believe that that area of such a square should be positive!

We also recall another fact about inequalities that is useful:

Fact 2. If $x > y$ and a is a positive real number, then $ax > ay$.

If b is a negative real number, then $bx < by$.

Question 4. What about division?

Answer 4. Division by a is just multiplication of $\frac{1}{a}$

Now that we know that we can multiply both sides of an inequality by the same real and read off a new inequality, the next logical question is: can we do it with the squaring function?

Theorem 5. *If* $a > b > 0$ *then* $a^2 > b^2$.

This property is called **monotonicity**. It says that if we square a bigger number than it is still bigger.

How can we prove this?

Proof. He we use the last result in an interesting way.

We have $a > b > 0$. In particular, $b > 0$ and $a > b$. Therefore, we can multiply both sides by b and get $ab > b^2$.

Similarly, we can multiply both sides by a and $a > 0$ and we get $a^2 > ab$.

Combining the two together, we get $a^2 > ab > b^2$. □

Question 5. Is the above true if just $a > b$?

Answer 5. No; Consider $a = 1$ and $b = -1,000,000$

2.4 Square Roots

Geometrically, x^2 represents the area of a square with side length x. Given the area, can we find the side length required?

Definition 3. For any $x \geq 0$, the unique $y \geq 0$ such that $y^2 = x$. y is called the **square root** of x, and is denoted $\sqrt{x}$.

The definition has a theorem inside of it that we really should prove. Why do positive numbers have square roots? Why are they unique? These questions require more about the properties of the real numbers than we wish to address right now.

We can however prove some interesting properties of the square root function, assuming it exists.

First, we prove a monotonicity result.

Theorem 6. *If $x > y$ then $\sqrt{x} > \sqrt{y}$*

Proof. Here, we are stuck. If we have $x > y$ how can we possible get $\sqrt{x} > \sqrt{y}$?

Well, we continue in this way. Suppose that we were wrong about this theorem being correct. Then we would have an x and y that **witness** this. They would be real numbers that contradict this monotonicity property. What does it mean to contradict this property?

Well, it would mean $x > y$ but $\sqrt{x} \leq \sqrt{y}$. Well, we know squaring is monotone over the positive reals, so if $\sqrt{x} \leq \sqrt{y}$ we can square both sides and see $x \leq y$. So we know $x > y$ and $x \leq y$. But these are opposite things!

Therefore, this theorem being wrong contradicts mathematics and causes the universe to explode! Therefore, since the universe didn't explode, and mathematics is sound, it must be the case that the theorem is correct. □

The above proof is called a **proof by contradiction**. We will study these in great detail later in the week.

2.5 Means

A mean is a way to combine a bunch of real numbers into some sort of real number which represents the value of all of them. There are two popular means.

Definition 4. The **arithmetic mean** of two real numbers x and y is $\frac{x+y}{2}$

The **geometric mean** of two non-negative real numbers x and y is $\sqrt{x \cdot y}$.

These two means both come up naturally in different applications. We won't actually talk about the applications. The applications to us is the following inequality:

Theorem 7 (AGM Inequality)**.** *If x and y are non-negative real numbers than the arithmetic mean of x and y is at least as large as the geometric mean; that is:*

$$\frac{x+y}{2} \geq \sqrt{x \cdot y}$$

Scratch Work. The proof is not long if one makes the correct observation. Making the correct observation without some thought and mistakes along the way is almost always impossible, however. Some things that would work are:

- If we were somehow able to prove that $\sqrt{x+y} \geq \sqrt{2}xy$, then we could use the monotonicity of squaring to square both sides.
- If we were somehow able to prove that $(x+y)^2 \geq 4xy$ then we could use the monotonicity of square roots.
- If we were to somehow able to prove that $x + 2\sqrt{xy} + y \geq 0$ then we could do some simple rearranging.

Now that we thought about some possible tactics, let's proceed!

It seems as though proving $(x+y)^2 \geq 4xy$ is the best tactic. We can see that $(x+y)^2 = x^2+2xy+y^2$. We want $x^2+2xy+y^2 \geq 4xy$. Well, let's group all the xy terms together, and we see we want $x^2-2xy+y^2 \geq 0$. This is factorable; this is $(x-y)^2 \geq 0$, which is a true statement!

Now, proving that something implies a true statement is meaningless. What we need is to rewrite this with the truth statement first, and then go backwards till we get what we want! Let's rewrite it.

Proof. Observe that $(x-y)^2 \geq 0$. Expanding, we see that $x^2-2xy+y^2 \geq 0$. If we add $4xy$ to both sides, we have that $x^2+2xy+y^2 \geq 4xy$. We can factor the left hand side and get $(x+y)^2 \geq 4xy$. Then, we can square root both sides by the monotonicity of square roots, and get $(x+y) \geq 2\sqrt{xy}$. Dividing both sides by 2, we get

$$\frac{x+y}{2} \geq \sqrt{xy}$$

Which is what we want! □

Problem 1. Let u, v, w be non-negative real numbers. Prove

$$uv + uw \geq u\sqrt{vw}$$

Solution 1. Use the AGM inequality on uv and uw, which are both nonnegative numbers. We get that

$$\frac{uv+uw}{2} \geq \sqrt{u^2vw} = u\sqrt{vw}$$

Multiply both sides by two, and get

$$uv + uw \geq 2u\sqrt{vw}$$

But, the multiplying by two only make the left hand side bigger! Thus, we can safely say that

$$uv + uw \geq u\sqrt{vw}$$

Question 6. Does multiplying by a number always make it bigger?

Answer 6. No! Raising to a power, multiplying numbers do not always making a number bigger!

- If you raise a number to an positive integer power, if the number is on $(0,1)$ it will make the number smaller! Think about the other cases.
- If you multiply a positive number by a number on $(0,1)$ is gets smaller! Think about the other cases.
- If you add a negative number, the number gets smaller.