Approximating Integrals

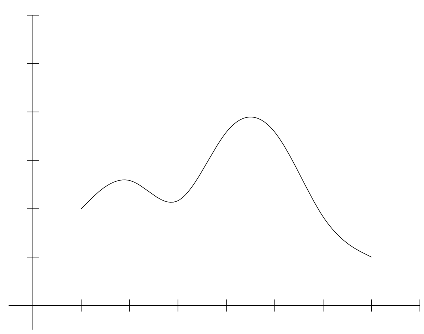
May 23, 2013

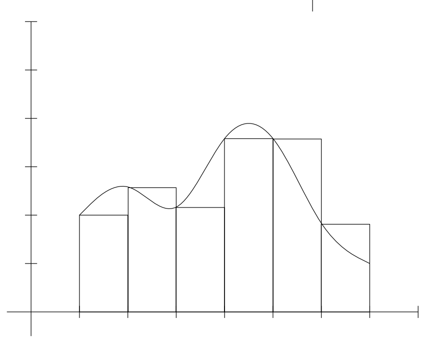
Goals:

- Review was the formal definition of the integral is.
- Use Riemann Sums to approximate an integral (left, right, midpoint).
- Use the Trapezoidal Rule and Simpson's Rule to approximate an integral.
- Talk about how bad these can be (error).

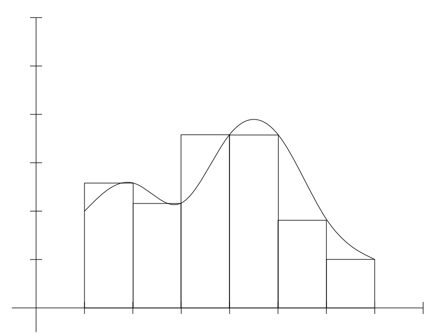
1 Review of the Definition of the Integral

Recall the definition for the integral $\int_a^b f(x)\ dx$. Although we've been talking about the integral as the anti-derivative recently (which is correct in most cases by the fundamental theorem of calculus), the integral itself is formally the limit of the left and right *Riemann Sums*.





Left Riemann Sum



Right Riemann Sum

$$R_L^n = \sum_{k=0}^{n-1} f(a + k \cdot \underbrace{\frac{b-a}{n}}_{\Delta x}) \cdot \underbrace{\frac{b-a}{n}}_{\Delta x}$$

$$R_R^n = \sum_{k=1}^{n} f(a + k \cdot \underbrace{\frac{b-a}{n}}_{\Delta x}) \cdot \underbrace{\frac{b-a}{n}}_{\Delta x}$$

So, we break the interval into n equally distributed regions of size $\Delta x = \frac{b-a}{n}$. Then for R_L^n we draw a rectangle in each interval with width Δx and height the height of the function on the left endpoint of the interval (as pictured). For R_R^n we do the same, but for the right endpoint. We define the integral to be:

$$\int_a^b f(x)\,dx := \lim_{n\to\infty} R_L^n$$
$$:= \lim_{n\to\infty} R_R^n$$

Note: If one of the limits above doesn't exist, or if they are not equal to each other, then we say the function is not *integrable*.

2 Motivation

We know how to take integrals of most sufficiently nice functions. Unfortunately, it is in principle impossible to write down a *closed form* solution to the integral of lots of very nice functions; a closed form is one written as the product, sum, exponent, and composition of all the functions that you know about. You may ask: well, is the ones we can find a closed form solution for good enough? The answer is no. For example:

$$\int_a^b \frac{1}{\sqrt{2\pi}} e^{\frac{x^2}{2}}\,dx$$

The integrand of the above is the function of the normal distribution with mean 0 and standard deviation 1. Therefore, the integral above calculates the probability that something drawn from that distribution will be between a and b. Unfortunately, the integral has no closed form solution.

Fortunately, we can calculate the above integral to arbitrary precision (just like we can calculate the integral of $\int \frac{1}{x}\,dx$ to arbitrary precision). Therefore, life isn't so bad. Today we'll learn how to calculate integrals like the above to some precision.

3 The Lazy Way: Riemann Sums

As the integral is defined in terms of the limit of Riemann sums, we can expect that the Riemann sums are a pretty good approximation of the function for large n.