

# Trigonometric Substitution

May 22, 2013

Goals:

- Do integrals using trigonometric substitution.

## 1 Motivation

We are motivated by the idea that when we do  $u$ -substitution with functions that involve trigonometric functions we often eliminate all trigonometric functions and end up with an easier looking function just involving powers, products, and sums. The idea is to do the opposite in order to integrate functions involving powers, products, and sums; that is, instead of doing a  $u$ -substitution, do an “inverse  $u$ -substitution” to add more complex functions (in particular: trigonometric functions). Then we can exploit different identities.

**Example 1.** The following isn't actually an example, but it is something about the motivation. Look at the following integral:

$$\int \sin(x) \cos(x) dx$$

Of course, one way is to do the easy substitution  $u = \sin(x)$  and one gets  $du = \cos(x) dx$ , which transforms the above integral to:

$$\int u du$$

Although this is an easy integral, let's pretend that we were having trouble with the integral involving  $u$  instead of the one involving  $x$ . Another way to integrate the first integral is to use a double angle formula:

$$\int \sin(x) \cos(x) dx = \frac{1}{2} \int \sin(2x) dx = -\frac{1}{2} \cos(2x) + C = \frac{1}{2} \sin^2(x) - \frac{1}{2} + C = \frac{1}{2} \sin^2(x) + C$$

The last two steps come from using a cosine double angle formula, and ‘aborbing’ the number into the constant. But remember our  $u$ -substitution; We have  $u = \sin(x)$ . Therefore, we have just shown, only using trigonometric integration, and without using the power rule, that

$$\int u du = \frac{1}{2} u^2 + C$$

## 2 When it's helpful

The above motivation is not a typical use for trigonometric substitution, but it does illustrate the point: we are doing a  $u$ -substitution in reverse and then exploiting trigonometric identities. The identity that we will exploit is:

$$\sin^2(x) + \cos^2(x) = 1$$

And the corresponding identity when you divide by  $\cos^2(x)$ :

$$\tan^2(x) + 1 = \sec^2(x)$$

Imagine with had an integral with the following things in it:

- $x^2 + a$

- $x^2 - a$
- $a - x^2$

If we undid a  $u$ -substitution for  $x$  being a particular trigonometric function, then we can use an identity to contract the sum into just one trigonometric function.

**Example 2.** To test our idea let's do one more integral that we could do by a standard  $u$ -substitution:

$$\int \frac{x}{\sqrt{1-x^2}} dx$$

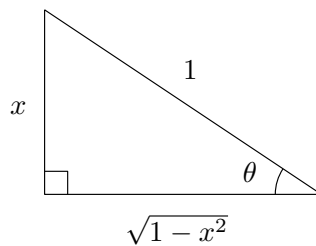
Here the part of our integral  $1 - x^2$  resembles  $1 - \sin^2(\theta)$ . Therefore, we'd have liked the above integral to come from a  $u$  substitution (or  $x$  substitution in this case) where  $x = \sin(\theta)$ . Then  $dx = \cos(\theta) d\theta$ . So we have

$$\int \frac{x}{\sqrt{1-x^2}} dx = \int \frac{\sin(\theta)}{\sqrt{1-\sin^2(\theta)}} \cos(\theta) d\theta$$

Using the identity that motivated the substitution and simplifying, we get:

$$\begin{aligned} \int \frac{\sin(\theta)}{\sqrt{1-\sin^2(\theta)}} \cos(\theta) d\theta &= \int \frac{\sin(\theta)}{\cos(\theta)} \cos(\theta) d\theta \\ &= \int \sin(\theta) d\theta \\ &= -\cos(\theta) + C \end{aligned}$$

It would be a little rude to submit this as an answer because it has  $\theta$ 's but our question was posed in terms of  $x$ . Therefore, we need to go back into terms of  $x$ . Our only rule is that  $x = \sin(\theta)$ . Therefore, the following triangle visualizes our information:



So,  $\cos(\theta) = \sqrt{1-x^2}$ . Therefore,

$$\int \frac{x}{\sqrt{1-x^2}} dx = -\sqrt{1-x^2} + C$$

**Example 3.**

$$\int \frac{1}{\sqrt{1-x^2}} dx$$

Note, this looks very similar to the last problem. Following the work of the above with the same substitution made, we arrive at this integral:

$$\int 1 d\theta$$

Obviously then, the answer is  $\theta$ . But how do we convert back to the land of  $x$ 's? We only know that  $x = \sin(\theta)$ ; therefore, given  $x$  one can find  $\theta$  by taking the arcsine of  $x$ :  $x = \arcsin(\theta)$ . So we have that

$$\int \frac{1}{\sqrt{1-x^2}} dx = \arcsin(x) + C$$

**Example 4.** We will do the following in class:

$$2 \cdot \int_{-1}^1 \sqrt{r^2 - x^2} dx$$

**Example 5.**

$$\int \sqrt{u^2 + 3} du$$

Here, the part of our integral  $u^2 + 3$  resembles  $\tan^2(\theta) + 1$ . In particular, we know that

$$(\sqrt{3} \tan(\theta))^2 + 3 = 3 \sec^2(\theta)$$

Therefore, we would like our  $u$  to have been obtained from a  $u$  substitution from where  $u = \sqrt{3} \tan(\theta)$ ; for this to have been a good substitution, we must have had  $du = \sqrt{3} \sec^2(\theta) d\theta$ . Therefore, we have:

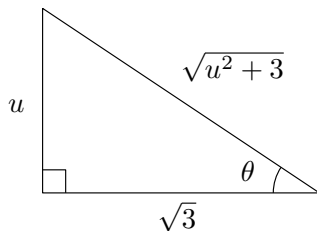
$$\int \sqrt{u^2 + 3} du = \int \left( \sqrt{3 \tan^2(\theta) + 3} \right) \left( \sqrt{3} \sec^2(\theta) \right) d\theta = 3 \int \sec^2(x) \sqrt{\tan^2(x) + 1} dx$$

Using the identity that motivated this substitution:

$$\begin{aligned} 3 \int \sec^2(x) \sqrt{\tan^2(x) + 1} dx &= 3 \int \sec^3(\theta) d\theta \\ &= 3 \left( \frac{1}{2} \sec(\theta) \tan(\theta) + \frac{1}{2} \int \sec(\theta) d\theta \right) \\ &= \frac{3}{2} \sec(\theta) \tan(\theta) + \frac{3}{2} \ln |\sec(\theta) + \tan(\theta)| + C \end{aligned}$$

Note: To integrate  $\sec^3(\theta)$ , I used the reduction formula we did yesterday.

This is not a very satisfying answer as the answer would be in terms of  $\theta$  but it was posed in terms of  $u$ . We need a way to convert between the two; the only rule we have connecting them is that  $u = \sqrt{3} \tan(\theta)$ ; or, slightly rephrased,  $\frac{u}{\sqrt{3}} = \tan(\theta)$ . The following triangle visualizes our information:



One can see then that  $\sec(\theta) = \frac{1}{\cos(\theta)} = \frac{\sqrt{u^2+3}}{\sqrt{3}}$ , and  $\tan(\theta) = \frac{u}{\sqrt{3}}$ . Therefore:

$$\frac{3}{2} \underbrace{\sec(\theta)}_{\frac{\sqrt{u^2+3}}{\sqrt{3}}} \underbrace{\tan(\theta)}_{\frac{u}{\sqrt{3}}} + \frac{3}{2} \ln \left| \underbrace{\sec(\theta)}_{\frac{\sqrt{u^2+3}}{\sqrt{3}}} + \underbrace{\tan(\theta)}_{\frac{u}{\sqrt{3}}} \right| = \frac{3}{2} \cdot \frac{u\sqrt{u^2+3}}{3} + \frac{3}{2} \ln \left( \frac{|\sqrt{u^2+3} + u|}{\sqrt{3}} \right)$$

Therefore, we have

$$\begin{aligned} \int \sqrt{u^2 + 3} du &= \frac{3}{2} \cdot \frac{u\sqrt{u^2+3}}{3} + \frac{3}{2} \ln \left( \frac{|\sqrt{u^2+3} + u|}{\sqrt{3}} \right) + C \\ &= \frac{u\sqrt{u^2+3}}{2} + \frac{3}{2} \ln |\sqrt{u^2+3} + u| - \underbrace{\frac{3}{2} \ln(\sqrt{3})}_{\text{constant}} + C \\ &= \frac{u\sqrt{u^2+3}}{2} + \frac{3}{2} \ln |\sqrt{u^2+3} + u| + C \end{aligned}$$

**Example 6.** We will do the following example in class:

$$\int \frac{dx}{\sqrt{9x^2 + 6x - 8}}$$

### 3 More Problems

1.  $\int \sqrt{1 - 4x^2} \, dx$

2.  $\int \frac{dt}{t^2 \sqrt{t^2 - 16}}$

3.  $\int_0^2 x^3 \sqrt{x^2 + 4} \, dx$

4.  $\int \frac{\sqrt{1 + x^2}}{x} \, dx$