Trigonometric Integrals

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Goals:

- Do integrals involving trigonometric functions.
- Review the derivatives for trigonometric functions.
- Review trigonometric identities

1 Trigonometric Derivatives

We first need to review the derivative rules for trigonometric functions. There are two which are the most important and come up the most:

$$\frac{d}{dx}\sin(x) = \cos(x) \qquad\qquad \frac{d}{dx}\cos(x) = -\sin(x)$$

But also:

$$\frac{d}{dx}\sec(x) = \sec(x)\tan(x) \qquad\qquad \frac{d}{dx}\tan(x) = \sec^2(x)$$

$$\frac{d}{dx}\csc(x) = -\csc(x)\cot(x) \qquad\qquad \frac{d}{dx}\cot(x) = -\csc^2(x)$$

2 Ad Hoc Integration

Given a function composed of some trig functions, one generally must perform adhoc techniques. In the next two section we deal with some very specific cases that tend to cover a lot of integrals one encounters due to trigonometric substitution (a technique we have not yet learned). The next techniques will also inspire what things may be necessary.

In general, converting all trigonometric function to sin's and cos's and breaking apart sums is not a terrible idea when confronted with a random integral. It may be easier, however, to view the problem in a different light (as is the case with integrals involving products of sec's and tan's).

3 Integration involving Sines and Cosines

If the function we are integrating is just a product of $\sin(x)$ and $\cos(x)$ our general strategy is the same: **change all $\sin$'s to $\cos$'s except for one, or vice versa**. We change sin's to cos's or cos's to sin's via Pythagorean's Theorem:

$$\sin^2(x) + \cos^2(x) = 1$$

Example 1.

$$\int \sin^3(x)\cos^2(x)\,dx$$

Note, writing $\sin^3(x)$ as $\sin(x)[\sin^2(x)]$ and then using the above theorem, we have:

$$\int \sin^3(x)\cos^2(x)\;dx = \int \left(1-\cos^2(x)\right)\cos^2(x)\sin(x)\;dx$$

Set $u=\cos(x)$; then $du=-\sin(x)\;dx$. Therefore, we have:

$$\int \left(1-\cos^2(x)\right)\cos^2(x)\sin(x)\;dx = \int \left(1-u^2\right)u^2\;du$$

Expanding and integrating:

$$\int \left(1-u^2\right)u^2\;du = \frac{u^3}{3}-\frac{u^5}{5}+C = \frac{\cos^3(x)}{3}-\frac{\cos^5(x)}{5}+C$$

When does this plan off attack fail for when we just have products of sin's and cos's? Well, if both of the powers of sin and cos are even, then we cannot 'save' one of them for the u-substitution. Here, we get creative with the following two rules:

$$\sin(2x)=2\sin(x)\cos(x) \qquad\qquad \cos(2x)=\cos^2(x)-\sin^2(x)$$

Using the Pythagorean's Theorem on the second we get two formulae:

$$\begin{aligned}\cos(2x)&=2\cos^2(x)-1\\ \cos(2x)&=1-2\sin^2(x)\end{aligned}$$

The idea: **Use the** sin **double angel formula as much as possible, and then with any 'left over'** sin**'s and** cos**'s use the** cos **double angle formula to convert everything in terms of** $\sin(2x)$ **and** $\cos(2x)$**. We can repeat this until one of the powers is odd.**

Example 2.

$$\int \sin^4(x)\cos^2(x)\;dx$$

Here, there is no easy way to make a substitution. Therefore, we use the double angle formulas.

$$\begin{aligned}\int \sin^4(x)\cos^2(x)\;dx &= \int (\sin(x)\cos(x))^2\sin^2(x)\;dx\\
&= \int \frac{1}{4}\left(2\sin(x)\cos(x)\right)^2\sin^2(x)\;dx\\
&= \int \frac{1}{4}\left(\sin(2x)\right)^2\frac{1}{2}\left(1-\cos(2x)\right)\;dx\\
&= \frac{1}{8}\int \left(\sin^2(2x)-\sin^2(2x)\cos(2x)\right)\;dx\\
&= \frac{1}{8}\left(\int \sin^2(2x)\;dx\right)-\frac{1}{8}\left(\int \sin^2(2x)\cos(2x)\;dx\right)\\
&= \frac{1}{8}\left(\int \frac{1}{2}\left(1-\cos(4x)\right)\;dx\right)-\frac{1}{8}\left(\int \sin^2(2x)\cos(2x)\;dx\right)\\
&= \frac{1}{16}\left(\int (1-\cos(4x))\;dx\right)-\frac{1}{8}\left(\int \sin^2(2x)\cos(2x)\;dx\right)\end{aligned}$$

And now, we just integrate; for the second integral, we do a u-substitution for $u=\sin(2x)$.

$$\begin{aligned}&= \frac{1}{16}\left(x-\frac{1}{4}\sin(4x)\right)-\frac{1}{8}\left(\frac{1}{6}\sin^3(2x)\right)+C\\
&= \frac{x}{16}-\frac{\sin(4x)}{64}-\frac{\sin^3(2x)}{48}+C\end{aligned}$$

4 Integration involving Secants and Tangents

The method for integrating some product of $\sec(x)$ and $\tan(x)$ is very similar to the above. As a general strategy one will want to do one of the following two things:

- Save a $\sec^2(x)$ and change everything else to $\tan(x)$.
- Save a $\sec(x)\tan(x)$ and change everything else to $\sec(x)$.

Here, the way we change between sec's and tan's is by Pythagorean's Theorem, just as above. If you take the formula $\sin^2(x) + \cos^2(x) = 1$ and divide entirely by $\cos^2(x)$ one gets:

$$\tan^2(x) + 1 = \sec^2(x)$$

One case see that in the case where you have an even (nonzero) power of $\sec(x)$ the first is possible. In the case where you have a odd power of $\tan(x)$ and at least one $\sec(x)$ then the second is possible.

Therefore, we are left with the three cases where the above heuristics don't work:

- What happens when you have a power of $\tan(x)$ and no $\sec(x)$?
- What happens when you have an even power of $\tan(x)$ and an odd power of $\sec(x)$?

The first case can be done (sort of) easily, but the second can be tricky. For the first, we prove a **reduction formula** for $\int \tan^n(x)\,dx$. Let us investigate what this integral is. First note, in the case when $n = 1$ this is easy as it is just $\ln|\sec(x)| + C$. If $n = 2$ this is a little trickier, but still not too difficult:

$$\int \tan^2(x)\,dx = \int (\sec^2(x) - 1)\,dx = \tan(x) - x + C$$

Therefore, we may assume that $n > 2$. Then we have

$$\begin{aligned}
\int \tan^n(x)\,dx &= \int \tan^2(x)\tan^{n-2}(x)\,dx \\
&= \int \big(\sec^2(x) - 1\big)\tan^{n-2}(x)\,dx \\
&= \int \sec^2(x)\tan^{n-2}(x)\,dx - \int \tan^{n-2}(x)\,dx \qquad\qquad u = \tan(x); du = \sec^2(x)\,dx \\
&= \frac{1}{n-1}\tan^{n-1}(x) - \int \tan^{n-2}(x)\,dx
\end{aligned}$$

Notice, now we have reduce the problem to an easier problem, since the power of tan is reduced by two. Eventually, by subtracting 2 over and over again, we are either integrating $\tan(x)$ or $tan^2(x)$. In fact, we can even use the reduction rule on $\tan^2(x)$ and reduce it to $\tan^0(x) = 1$.

Example 3.

$$\int \tan^6(x)\,dx$$

Use the reduction formula:

$$\begin{aligned}
\int \tan^6(x)\,dx &= \frac{1}{5}\tan^5(x) - \left(\int \tan^4(x)\,dx\right) \\
&= \frac{1}{5}\tan^5(x) - \frac{1}{3}\tan^3(x) + \left(\int \tan^2(x)\,dx\right) \\
&= \frac{1}{5}\tan^5(x) - \frac{1}{3}\tan^3(x) + \tan(x) - \left(\int dx\right) \\
&= \frac{1}{5}\tan^5(x) - \frac{1}{3}\tan^3(x) + \tan(x) - x + C
\end{aligned}$$

In the case where there is an even power of tangent and an odd power of secant, there are different approaches. Often one can be creative and find nice trigonometric formulas to use to simplify the problem. In general though, one can always integrate by changing all the tan's to sec's; this reduces the problem to being able to integrate things of the form $\sec^n(x)$.

$$\begin{aligned}
\int \sec^n(x) &= \int \sec^2(x)\sec^{n-2}(x)\ dx \\
&\qquad u = \sec^{n-2}(x); dv = \sec^2(x)\ dx \\
&\qquad du = (n-2)\sec^{n-2}(x)\tan(x)\ dx; v = \tan(x) \\
&= \sec^{n-2}(x)\tan(x) - (n-2)\int \sec^{n-2}(x)\tan^2(x)\ dx \\
&= \sec^{n-2}(x)\tan(x) - (n-2)\int \sec^{n-2}(x)(\sec^2(x)-1)\ dx \\
&= \sec^{n-2}(x)\tan(x) - \left((n-2)\int \sec^n(x)\ dx\right) + \left((n-2)\int \sec^{n-2}(x)\ dx\right) \\
&= \frac{1}{n-1}\sec^{n-2}(x)\tan(x) + \frac{n-2}{n-1}\int \sec^{n-2}(x)\ dx
\end{aligned}$$

The last step comes from adding the middle integral to both sides, and then dividing by $n+1$. Now, since we are reducing the power by 2 each time, we may be left with integrating $\sec(x)$ at the end or just integrating 1. In the case when we are integrating 1, obviously we are done; therefore we can integrate all even powers of $\sec(x)$ by using the reduction rule. Note however, that we didn't need to use the reduction rule for this case; this is in the case discussed above with an even power of $\sec(x)$, which is always easy.

So, the reduction rule above is incomplete because we do not know how to integrate $\sec(x)$. We will take the following as a rule that can be quickly checked using the rules for differentiation:

$$\int \sec(x)\ dx = \ln|\sec(x)+\tan(x)|$$

There is essentially no way around memorizing the above.

Example 4.

$$\int \sec^3(x)\tan^2(x)\ dx$$

Notice, there are no easy substitutions. Therefore, we will change everything to sec's and use the reduction formula.

$$\begin{aligned}
\int \sec^3(x)\tan^2(x)\ dx &= \int \left(\sec^5(x) - \sec^3(x)\right)\ dx \\
&= \int \sec^5(x)\ dx - \int \sec^3(x)\ dx \\
&= \frac{1}{4}\sec^3(x)\tan(x) + \frac{3}{4}\int \sec^3(x)\ dx - \int \sec^3(x)\ dx \\
&= \frac{1}{4}\sec^3(x)\tan(x) - \frac{1}{4}\int \sec^3(x)\ dx \\
&= \frac{1}{4}\sec^3(x)\tan(x) - \frac{1}{4}\left(\frac{1}{2}\sec(x)\tan(x) + \frac{1}{2}\int \sec(x)\ dx\right) \\
&= \frac{1}{4}\sec^3(x)\tan(x) - \frac{1}{8}\sec(x)\tan(x) - \frac{1}{8}\ln|\sec(x)+\tan(x)| + C
\end{aligned}$$

Note: Everything you read in this section can be applied to integrating products of cot's and csc's.

The above formulas do not need to be memorized, but you may be asked to derive one of these formulas on a quiz. So please know the general strategy for deriving a reduction formula like the above.

5 Scaled Angles

What we talked about was integrating things of the form $\sin^m(sx)\cos^n(sx)$ where s is any real number. What if the scalar is different in each function? Here, we must remember the sum formulas:

$$\sin(\alpha+\beta) = \sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta)$$
$$\cos(\alpha+\beta) = \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)$$

Note: The way I remember these is just by remembering the general form, and making sure they match with the double angles when you do $\sin(2x) = \sin(x+x)$.

In particular, we have the following two equations:

$$\sin(\alpha+\beta) = \sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta)$$
$$\sin(\alpha-\beta) = \sin(\alpha)\cos(\beta) - \cos(\alpha)\sin(\beta)$$

Here, we are using the cos is an even function and sin is an odd function. Adding the two, one gets:

$$\sin(\alpha)\cos(\beta) = \frac{1}{2}\left(\sin(\alpha+\beta) + \sin(\alpha-\beta)\right)$$

Doing a similar thing to the cos formula, one gets rules that will help for integrals of the form $\cos(\alpha)\cos(\beta)$ and $\sin(\alpha)\sin(\beta)$.

Example 5.

$$\int \sin(2x)\cos(5x)\;dx$$

Here, we use the sum formulas:

$$\begin{aligned}\int \sin(2x)\cos(5x)\;dx &= \frac{1}{2}\int \left(\sin(7x) + \sin(-3x)\right)\;dx \\ &= \frac{1}{2}\left(-\frac{1}{7}\cos(7x) + \frac{1}{3}\cos(-3x)\right) + C \\ &= \frac{1}{6}\cos(3x) - \frac{1}{14}\cos(7x) + C\end{aligned}$$

Note, because cos is even, $\cos(-3x) = \cos(3x)$.