Some things we have to know from precalculus:

(1) Algebra

(a) Understand functions The function is the most important idea in mathematics. A function is a machine that that's in a value and gives an output depending on that value.

In this class, functions will be rules, and passing a value into a function will be substituting it in for the variable.

Example: $f(x) = x^2 + 2x + 1$.

Here, we can view f as a machine that takes in $\square$ and returns $\square^2 + 2\square + 1$. We can put any number (or algebraic expression representing a number) into that box. So,

$$\begin{aligned} f(1) &= 1^2 + 2(1) + 1 = 4 \\ f(\pi) &= \pi^2 + 2\pi + 1 \\ f(h+n) &= (h+n)^2 + 2(h+n) + 1 \end{aligned}$$

Example: $f(x) = \dfrac{2x+1}{x+3}$

$$\begin{aligned} f(1) &= \frac{2(1)+1}{1+3} = \frac{3}{4} \\ f(x+h) &= \frac{2(x+h)+1}{(x+h)+3} \\ f(\frac{y+1}{2}) &= \frac{2\left(\frac{y+1}{2}\right)+1}{\left(\frac{y+1}{2}\right)+3} \end{aligned}$$

(b) Manipulating fractions Manipulating fractions is an extremely important part of the a good mathematical background. You should know how to add two fractions. To add $\frac{a}{b} + \frac{c}{d}$ you need to find a common denominator; here is is bd. Then one can multiply $\frac{d}{d}$ on the first fraction and $\frac{b}{b}$ on the second (which is just 1 so it doesn't change the value) and we get

$$\frac{ad}{bd} + \frac{cb}{bd}$$

But these have the same denominator, so we can add:

$$\frac{ad+cb}{bd}$$

Example: Simplify $\frac{x}{x^2+1} + \frac{x^2+x}{2x+1}$

Solution: a common denominator is $(x^2+1)(2x+1)$. Thus we multiply top and bottom of the first fraction by $2x+1$ and of the second by x^2+1 to get both fractions in terms of this common denominator.

$$\begin{aligned} \frac{x}{x^2+1} + \frac{x^2+x}{2x+1} &= \left(\frac{2x+1}{2x+1}\right)\frac{x}{x^2+1} + \left(\frac{x^2+1}{x^2+1}\right)\frac{x^2+x}{2x+1} \\ &= \frac{x(2x+1)}{(x^2+1)(2x+1)} + \frac{(x^2+x)(x^2+1)}{(x^2+1)(2x+1)} \end{aligned}$$

Now they have a common denominator, so we can just add:

$$\frac{x(2x+1)}{(x^2+1)(2x+1)} + \frac{(x^2+x)(x^2+1)}{(x^2+1)(2x+1)} = \frac{x(2x+1) + (x^2+x)(x^2+1)}{(x^2+1)(2x+1)}$$

Depending on what you want to do with this fraction, this may be as simple as you need. You can fact an x out of the numerator:

$$\frac{x\left(2x+1+(x+1)(x^2+1)\right)}{(x^2+1)(2x+1)}$$

and expand the top a bit:

$$\frac{x\left(2x+1+x^3+x^2+x+1\right)}{(x^2+1)(2x+1)} = \frac{x\left(x^3+x^2+3x+2\right)}{(x^2+1)(2x+1)}$$

We can also break apart sums (or differences) over fractions, that is $\frac{a+b}{c} = \frac{a}{c} + \frac{b}{c}$.

Example: Expand $\frac{x^2+2}{x^2}$

Solution:

$$\begin{aligned}\frac{x^2+1}{x^2} &= \frac{x^2}{x^2} + \frac{1}{x^2}\\ &= 1 + \frac{1}{x^2}\end{aligned}$$

Because products and quotients work different over fractions of course (since fractions are quotients). One can take a fraction like $\frac{ab}{c}$ and can factor out the a or the c or both!

$$\frac{ab}{c} = a\frac{b}{c} = b\frac{a}{c} = ab\frac{1}{c}$$

Because of this we can "cancel" terms on the top and bottom of fractions when there is an entire factor of that term on both sides.

Example: Simplify $\frac{x^2+x}{x}$ Solution:

$$\begin{aligned}\frac{x^2+x}{x} &= \frac{x(x+1)}{x}\\ &= (x+1)\frac{x}{x}\\ &= x+1\end{aligned}$$

Quotients (division) within fractions are ugly, as they are "fractions within fractions." Whenever we see them, we often try to get rid of them quickly but multiplying by the common denominator of the fractions within the fractions to the top and the bottom. It's best to exhibit this with an example:

Example: Simplify $\frac{\frac{x}{2x+1}+x^2}{\frac{x^2+1}{x}+\frac{x+1}{x^3}}$

Solution: One can see that a common denominator of all the fractions within the fraction is $(2x+1)(x)(x^3)$. So we multiply the top and the bottom of the fraction by that

$$\begin{aligned}\frac{\frac{x}{2x+1}+x^2}{\frac{x^2+1}{x}+\frac{x+1}{x^3}} &= \frac{\frac{x}{2x+1}+x^2}{\frac{x^2+1}{x}+\frac{x+1}{x^3}} \cdot \left(\frac{(2x+1)(x)(x^3)}{(2x+1)(x)(x^3)}\right)\\ &= \frac{(\frac{x}{2x+1}+x^2)(2x+1)(x)(x^3)}{(\frac{x^2+1}{x}+\frac{x+1}{x^3})(2x+1)(x)(x^3)}\\ &= \frac{\frac{x(2x+1)(x)(x^3)}{2x+1}+x^2(2x+1)(x)(x^3)}{\frac{(x^2+1)(2x+1)(x)(x^3)}{x}+\frac{(x+1)(2x+1)(x)(x^3)}{x^3}}\\ &= \frac{x(x)(x^3)+x^2(2x+1)(x)(x^3)}{(x^2+1)(2x+1)(x^3)+(x+1)(2x+1)(x)}\\ &= \frac{x^5+x^6(2x+1)}{x^3(2x+1)(x^2+1)+x(x+1)(2x+1)}\end{aligned}$$

(c) Solving Equations You should be able to solve equations of one variables, especially find the roots of polynomials, etc. From what's I've seen, particular attention should be paid on dealing with quotients.

Example: Solve $\frac{x+1}{x+2} = 3$ for x

Solution:

$$\begin{aligned}
&\frac{x+1}{x+2} = 3 \\
\implies &x+1 = 3(x+2) \\
\implies &x+1 = 3x+6 \\
\implies &-2x = 5 \\
\implies &x = \frac{-5}{2}
\end{aligned}$$

Example: Solve $\frac{x}{x+2} + \frac{1}{x+1} = 7$

Solution: we begin by finding a common denominator for the two terms. It is $(x+2)(x+1)$. Then we multiply both sides of the equation by that, which will essentially "get rid of" all the fractions.

$$\begin{aligned}
&\frac{x}{x+2} + \frac{1}{x+1} = 2 \\
\implies &x(x+1) + x + 2 = 2(x+1)(x+2) \\
\implies &(x^2+x) + (x+2) = 2(x^2+3x+2) \\
\implies &x^2+2x+2 = 2x^2+6x+4 \\
\implies &x^2+4x+2 = 0
\end{aligned}$$

Using the quadratic formula, one sees that $x = \frac{-4\pm\sqrt{16-4(2)}}{2}$, or $x = \frac{-4\pm\sqrt{8}}{2}$. We can simplify this further by observing $\sqrt{8} = \sqrt{4\cdot 2} = \sqrt{4}\sqrt{2} = 2\sqrt{2}$, and then canceling the 2's we get $-2\pm\sqrt{2}$. This means there are two answer: $-2+\sqrt{2}$ and $-2-\sqrt{2}$.

(d) Equations of lines

Given a point and the slope of a line, one can determine the equation of the line.

If the slop of a line is m, and a point on a line is (x_1, y_1) then the equation of the line is given by

$$y - y_1 = m(x - x_1)$$

Depending what you want to do with this, other simplifications may be more useful.

Given two points, one can find the line going through both of those point. Say the points are (x_1, y_1) and (x_2, y_2). Then the slope of the line is given by

$$m = \frac{y_1 - y_2}{x_1 - x_2}$$

ie. it is the change in y divided by the change in x.

Factoring

Very often, one would like to write things as a product of terms instead of a sum. The process of turning a sum into a product is called factoring. The opposite process of turning a product into a sum is called expanding.

Factoring takes advantage of the distributive property of addition, that is $a(b+c) = ab+ac$. As written, we brought the a into the parentheses, expanding the product into the sum. Factoring is the opposite process, ie. bringing the a out of the common terms.

Why do we want to factor? Many times it's to cancel terms. From the fractions section, you see that working with a fraction such as $\frac{ab}{a}$ is nice because you can cancel the a's and you just get b. You can do this because you can factor an a out of the denominator.

Let's begin with some simple factoring. We'll factor some quadratic expressions. To first see how to do this. Any quadratic with real roots can be written as:

$$a(x+r_1)(x+r_2)$$

where $-r_1$ and $-r_2$ are the roots of the quadratic. Expanding this, we get

$$a(x^2 + (r_1+r_2)x + r_1 r_2)$$

Thus, to factor something of the form $x^2 + bx + c$ we must find two numbers r_1 and r_2 who sum up to b and who multiply to c

Example: Factor $x^2 - 2x + 1$

Solution: Here, we can factor in our head. We must find two numbers that multiply to be 1 and who add to be -2. Here the choice is clear: -1 and -1. Thus, $x^2 - 2x + 1 = (x - 1)(x - 1) = (x - 1)^2$.

Example: Factor $x^2 - 7x - 8$

Solution: Here we must find two numbers who add together to -7 but who multiply together to -8. As we need them to multiply together to a negative number, this tells us one must be positive and the other negative. Thus the answer is -8 and 1 and $1 - 8 = -7$. So, $x^2 - 7x - 8 = (x - 8)(x + 1)$.

Example: Factor $x^2 - 16$.

Solution: Here we have to find two numbers who multiply together to be -16 and who sum to be 0. Thus one is positive, and the other negative, and they must be the same number in absolute values. Thus it bust be $\sqrt{16} = 4$. Thus, $x^2 - 16 = (x + 4)(x - 4)$.

In general, one can factor $x^2 - a$ where $a > 0$ as $(x + \sqrt{a})(x - \sqrt{a})$

Sometimes, we can also factor cubics.

Example: Factor $x^3 - x^2 + x - 1$

Solution: This is a trick that sometimes work. We can take a x^2 out of the first two terms, and get

$$\begin{aligned} x^3 - x^2 + x - 1 &= x^2(x - 1) + x - 1 \\ &= x^2(x - 1) + (x - 1) \end{aligned}$$

Note, we can factor out an $x - 1$.

$$= (x - 1)(x^2 + 1)$$

Thus, $x^3 - x^2 + x - 1 = (x - 1)(x^2 + 1)$.

This trick doesn't help us too much. If we know the root of the expression (which we often do), it becomes easier as we can do long division to take out the root. It's best to exhibit this with an example.

Example: Factor $x^3 - 3x^2 + x + 2$. Hint: When $x = 2$ this expression evaluates to 0.

Solution: We can perform long division. The hint tells us that 2 is a root, thus there must be a way to "divide out" the factor $(x - 2)$ (because that is a linear factor which is 0 when $x = 2$). So we do the division algorithm:

$$\begin{array}{r|rrr} & x^2 & -x & -1 \\ \hline x-2 & x^3 - 3x^2 & +x & +2 \\ & -x^3 + 2x^2 & & \\ \hline & -x^2 & +x & \\ & x^2 & -2x & \\ \hline & & -x & +2 \\ & & x & -2 \\ \hline & & & 0 \end{array} \tag{1}$$

Thus we get that $\frac{x^3-3x^2+x+2}{x-2} = x^2 - x - 1$. Written a different way,

$$x^3 - 3x^2 + x + 2 = (x^2 - x - 1)(x - 2)$$

The quadratic formula tells us that a quadratic in the form $ax^2 + bx + c$ has real roots if and only if $b^2 - 4ac \geq 0$. Looking at that in this case on $x^2 - x - 1$, one sees that the quadratic does not have real roots, and thus is not factorable. Sp this is the simplest we can factor.

(2) Trig

(a) Unit Circle You need to know the unit circle, and all the special trig values.

Angle (deg)	Angle (rad)	Sine	Cosine	Tangent
0	0	0	1	0
30^o	$\pi/6$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{3}}$
45^o	$\pi/4$	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	1
60^o	$\pi/3$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$
90^o	$\pi/2$	1	0	undef

and in other quadrants.

(b) Trig identities At some point in the course, we will make use of the following identities

- $\sin^2(x) + \cos^2(x) = 1$
- $\sin(2x) = 2\sin(x)\cos(x)$
- $\cos(2x) = \cos^2(x) - \sin^2(x)$