4.2.11 \[ \min \begin{align*} y_1 + 3y_2 + 8y_3 \\ y_1 + y_2 + y_3 \geq 12 \\ 5y_1 - 2y_2 + 4y_3 \leq 20 \\ 2y_1 + 3y_2 - y_3 = 12 \\ y_1 \geq 0, \ y_2 \geq 0, \ y_3 \text{ unrestricted.} \end{align*} \]

The dual problem has 3 constraints, constraints 1 and 2 are \( \leq \) since \( y_1, y_2 \geq 0 \) and 3rd constraint is \( = \) since \( y_3 \) is unrestricted.

Thus the dual is

\[ \max \begin{align*} 12x_1 + 20x_2 + 12x_3 \\ x_1 + 5x_2 + 2x_3 \leq 4 \\ x_1 - 2x_2 + 3x_3 \leq 3 \\ x_1 + 4x_2 - x_3 = 8 \\ x_1 \geq 0, \ x_2 \leq 0, \ x_3 \text{ unrestricted.} \end{align*} \]

4.2.12 \[ \min \begin{align*} 24y_1 + 33y_2 + 60y_3 \\ y_1 + y_2 + 4y_3 = 85 \\ 3y_1 + 4y_2 + y_3 \geq 35 \\ y_1 + y_2 + 4y_3 \geq 70 \\ 5y_1 - 2y_2 + 3y_3 \geq 45 \\ y_1 \geq 0, \ y_2 \text{ unrestricted, } y_3 > 0 \end{align*} \]

Dual has 3 constraints, 1 and 3 are \( \leq \) and 2nd is \( = \) since \( y_2 \) unrestricted.

Thus the dual is

\[ \max \begin{align*} 85x_1 + 35x_2 + 70x_3 + 4x_4 \\ 4x_1 + 3x_2 + x_3 + 5x_4 \leq 24 \\ x_1 + 4x_2 + x_3 - 2x_4 = 33 \\ 4x_1 + x_2 + x_3 + 3x_4 \leq 60 \\ x_1 \text{ unrestricted, } x_4 \geq 0, \ x_3, x_2 \geq 0 \end{align*} \]
max \[ 2x_1 + x_2 \]
\[ x_1 - x_2 \leq 12 \]
\[ x_1 \leq 14 \]
\[ x_2 \geq 0, x_1 \geq 0 \]

dual is
min \[ 12y_1 + 14y_2 \]
\[ y_1 + y_2 \geq 2 \]
\[ -y_1 \geq 1 \]
\[ y_2 \geq 0 \]
\[ y_1 \geq 0, y_2 \geq 0 \]

Note that since \( y_1 \geq 0 \) then \(-y_1 \leq 0\) so no \( y_2 \geq 0 \) satisfies \(-y_1 \geq 1\), thus the dual has no solution.

max \[ 5x_1 + 4x_2 + 9x_4 \]
\[ x_1 - x_2 + 3x_3 + 3x_4 = 4 \]
\[ x_1 + x_2 + x_3 + 10x_4 \leq 1 \]
\[ x_1 \leq 0, x_2 \leq 0, x_3 \geq 0, x_4 \geq 0 \]

We formulate the dual, solve the dual graphically and recover the \( x's \) using complementary slackness.

The dual is
min \[ 4y_1 + y_2 \]
\[ y_1 + y_2 \leq 5 \]
\[ -y_1 + y_2 \leq 4 \]
\[ 3y_1 + y_2 \geq 0 \]
\[ 3y_1 + 10y_2 \geq 9 \]
\[ y_1 \text{ unrestricted, } y_2 \geq 0 \]

Plotting we see

Objective equals 20.

Opt point \( Z = 20 \).
So the optimal solution is \( y_1 = 5, y_2 = 0 \).

Notice that the constraints 2, 3 and 4 in the dual are not tight, so \( x_2 = 0, x_3 = 0, x_4 = 0 \), since \( y_1 = 5 \) we look at the constant

\[(y - x_1 + x_2 - 3x_3 - 3x_4) \cdot y_1 = 0 \quad \text{which implies} \quad y - x_1 = 0 \quad \text{so} \quad x_1 = y.

Thus the opt. solution is \( x_i = y, x_2 = x_3 = x_4 = 0 \).

### 4.3.4

<table>
<thead>
<tr>
<th>( x_1 )</th>
<th>( x_2 )</th>
<th>( x_3 )</th>
<th>( s_1 )</th>
<th>( s_2 )</th>
<th>( s_3 )</th>
<th>( s_4 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1/5</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1/2</td>
<td>0</td>
<td>-2</td>
</tr>
<tr>
<td>5</td>
<td>1</td>
<td>0</td>
<td>-2</td>
<td>0</td>
<td>5/2</td>
<td>225</td>
</tr>
<tr>
<td>-1</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1/2</td>
<td>0</td>
<td>125</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1/2</td>
<td>0</td>
<td>125</td>
</tr>
<tr>
<td>2</td>
<td>0</td>
<td>0</td>
<td>-5/2</td>
<td>1</td>
<td>-5/2</td>
<td>65</td>
</tr>
<tr>
<td>12</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>40</td>
<td>0</td>
<td>25</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>( S_1 )</th>
<th>( S_2 )</th>
<th>( S_3 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>84</td>
<td>225</td>
<td>125</td>
</tr>
<tr>
<td>65</td>
<td>125</td>
<td>65</td>
</tr>
</tbody>
</table>

\( \begin{pmatrix} S_1 \\ S_2 \\ S_3 \end{pmatrix} = \begin{pmatrix} 84 \\ 225 \\ 125 \end{pmatrix} + \delta \begin{pmatrix} 1/4 \\ -2 \\ 1/3 \end{pmatrix} \)

Since \( S_3 \) needs to be \( 20 \) in all coordinates

we need \( 125 - 25 \cdot 20 \Rightarrow \delta \leq 12.5 \)

and \( 65 - 5/2 \cdot \delta \leq 0 \Rightarrow \delta \leq 26 \)

So maximum increase is \( \delta = 26 \) which gives a profit increase of \( 26 \cdot 40 = 1040 \)

dual variable \( x_2 \) increases.

For 3rd constraint

\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 84 \\ 225 \\ 125 \end{pmatrix} + \delta \begin{pmatrix} -2 \\ 1/2 \\ -1/2 \end{pmatrix} \)

so we need \( 84 - 2\delta \geq 0 \) and \( 65 - 5/2 \delta \geq 0 \)

\( \Rightarrow \delta \leq 42 \) and \( \delta \leq 150/3 = 50 \frac{1}{3} \)

So max increase is \( \delta = 50 \frac{1}{3} \) and profit increase is \( 40 \cdot 25 = 1050 \), larger than

for 2nd constraint.

b) The dual problem has 4 variables. Due to every constraint and

\( y_1 = 0, y_2 = 40, y_3 = 0, y_4 = 25 \)

c) \( x_2 \) is in the basis, changing the obj. func. coeff by \( \delta \) changes the obj. row.

We need obj. coeff for non-basics to be \( \geq 0 \), the values for non-basics are

\[ (12, 40, 25) + \delta (-1, -2, -5/2) = (12 - \delta, 40 - 2\delta, 25 + 5\delta) \]

So we need

\[ 12 - \delta \geq 0 \quad 40 - 2\delta \geq 0 \quad 25 + 5\delta \geq 0 \]

\( \Rightarrow -10 \leq \delta \leq 12 \) If \( \delta \) satisfies this then the basis is still optimal.
The LP's

\[
\begin{align*}
\text{max } & \quad 30x_1 + 100x_2 + 0.64x_3 - 0.66x_4 \\
& \quad x_1 + x_2 \\
& \quad x_3 + 4x_4 \\
& \quad 10x_1 + 20x_2 + x_3 - x_4 \leq 100 \\
& \quad x_4 \leq 160 \\
& \quad x_4 \leq 1100 \\
& \quad x_1, \ldots, x_4 \geq 0
\end{align*}
\]

and the opt. tableau

| \(x_1\) | \(x_2\) | \(x_3\) | \(x_4\) | \(s_1\) | \(s_2\) | \(s_3\) | \(s_4\) | \\|---|---|---|---|---|---|---|---| \\| 0 | 0 | -1 | 1 | -666.7 | 5.333 | -1 | 0 | 100 \\
| 0 | 1 | 0 | 0 | -0.333 | 0.111 | 0 | 0 | 20 \\
| 1 | 0 | 0 | 0 | 1.111 | -0.333 | 0 | 0 | 80 \\
| 0 | 0 | 1 | 0 | -1.222 | 0.722 | 1 | 400 | 437.7 \\
| 0 | 0 | 0.02 | 0 | 6.257 | 23.117 | 0.058 | 4.377 | \\|---|---|---|---|---|---|---|---| 

\(a)\) If we change the obj. fun. coeff. of \(x_4\) by \(\delta\), we need
\[(0.02, 6.257, 23.117, 0.058) + \delta(-1, 6.667, 3.333, -1)\]
\(
\begin{pmatrix}
(\delta) \\
(x_4) \\
(\delta)
\end{pmatrix}
\)
to be \(\geq 0\) in all coordinates.

This implies
\[
-6.257 \leq \delta \leq 0.02,
\]
\[
-6.257 \leq \delta \leq 0.02,
\]
thus the basis is opt. as long as obj. coeff. of \(x_4\) is in
\[
[-6.257, 0.02].
\]

\(b)\) and \(c)\) We change the RHS of \(3^{rd}\) constraint by \(\delta\) then we need
\[
\begin{pmatrix}
x_4 \\
x_3 \\
x_2 \\
x_1
\end{pmatrix} = \begin{pmatrix}
100 \\
20 \\
\delta \\
\delta
\end{pmatrix} + \delta \begin{pmatrix}
0 \\
0 \\
0 \\
1
\end{pmatrix} = \begin{pmatrix}
100+\delta \\
20 \\
\delta \\
\delta+\delta
\end{pmatrix}
to be \(\geq 0\). This implies
\[
-900 \leq \delta \leq 100.
\]

Since \(\delta = 0.02\) we get the opt. rule by plugging in \(\delta = 50\)

\[
\begin{pmatrix}
x_4 \\
x_3 \\
x_2 \\
x_1
\end{pmatrix} = \begin{pmatrix}
50 \\
20 \\
80 \\
450
\end{pmatrix}
\]
all other \(s\) equal 0.

The basis is opt. as long as the RHS of \(3^{rd}\) constraint is between 700 and 1200.

If the RHS is 700 or 1200, one of the basic variables is 0 and the solution is degenerate.

\(d)\) Since \(x_3\) is non-basic the price would need to increase by 0.02 for it to be profitable for \(x_3\) > 0, i.e. the increase should equal the dual (shadow) variable for \(x_3\) which has a value of 0.02.
\[ \text{Max } 50x_1 + 40x_2 \]

\[ \begin{align*}
x_1 + x_2 & \leq 50 \quad \text{(acres)} \\
3x_1 + 2x_2 & \leq 120 \quad \text{(days)} \\
10x_1 + 60x_2 & \leq 1200 \quad \text{(cows)} \\
20x_1 + 10x_2 & \leq 800 \quad \text{(fertilizer)} \\
x_1, x_2 & \geq 0
\end{align*} \]

**Optimal Tableau**

<table>
<thead>
<tr>
<th>(x_1, x_2, s_1, s_2, s_3, s_4)</th>
<th>( \alpha )</th>
<th>(x_1 = 30 \text{ acres})</th>
<th>(x_2 = 15 \text{ acres})</th>
<th>(s_1 = 5 \text{ acres (leftover)})</th>
<th>(s_2 = 0 \text{ days})</th>
<th>(s_3 = 0 \text{ $})</th>
<th>(s_4 = 50 \text{ pounds of fertilizer leftover})</th>
</tr>
</thead>
<tbody>
<tr>
<td>0 0 1 1/12 1/10 0 0 5</td>
<td>(a)</td>
<td>(0 1 0 1/12 2/10 0 0 15)</td>
<td>(0 0 0 1 5/8 1/7 0 50)</td>
<td>(0 0 0 1/5 1/9 0 0 30)</td>
<td>(0 0 0 65/9 1/8 0 0 2100)</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

b) An additional day of labour has \(\frac{65}{9}\) value \((\text{obj. coeff of } s_2)\).

c) Change RHS of constraint 3 by \(\delta\), we need

\[
\begin{pmatrix} s_1 \\ x_2 \\ s_3 \\ x_1 \end{pmatrix} = \begin{pmatrix} 5 \\ 15 \\ 50 \\ 30 \end{pmatrix} + \delta \begin{pmatrix} -1/160 \\ 3/160 \\ 16/160 \\ -1/80 \end{pmatrix} = \begin{pmatrix} 50 - \delta/160 \\ 15 + \delta/160 \\ 50 + \delta/160 \\ 30 - \delta/80 \end{pmatrix}
\]

so we need

\[
\begin{align*}
\delta & \leq 50 \cdot 160 = 8000 \\
\delta & \geq -15 \cdot 160 = -2400 \\
\delta & \geq -16 \cdot 50 = -800 \\
\delta & \leq 30 \cdot 80 = 2400
\end{align*}
\]

so \(-800 \leq \delta \leq 2400\), so the capital needs to be between 400 and 3600 to lean the same basis optimal.

d) plug in \(\delta = 160\) and get \(\begin{pmatrix} s_1 \\ x_2 \\ s_3 \\ x_1 \end{pmatrix} = \begin{pmatrix} 49 \\ 18 \\ 60 \\ 28 \end{pmatrix}\) so \(x_1 = 28, x_2 = 18, s_1 = 49, s_2 = 0, s_3 = 0, s_4 = 60\).

e) Let \(x_3\) be \# of days of additional labour.

\[ \text{Max } 50x_1 + 40x_2 \]

\[ \begin{align*}
x_1 + x_2 & \leq 50 \quad \text{(acres)} \\
3x_1 + 2x_2 & \leq 120 \quad \text{(labour)} \\
10x_1 + 60x_2 & \leq 1200 \quad \text{(cows)} \\
25x_3 & \leq 500 \quad \text{(wages (labour $))} \\
20x_1 + 10x_2 & \leq 800 \quad \text{(fertilizer)} \\
x_1, x_2, x_3 & \geq 0
\end{align*} \]
\[ \text{max} \quad 80x_1 + 120x_2 + 140x_3 + 160x_4 \\
\quad x_3 + x_4 = 2700 \\
2x_1 + 3x_2 + 4x_3 + 7x_4 \leq 9200 \\
3x_1 + 4x_2 + 5x_3 + 6x_4 \leq 10000 \\
\quad x_1, \ldots, x_4 \geq 0 \]

**Optimal Tableau**

<table>
<thead>
<tr>
<th></th>
<th>(x_1)</th>
<th>(x_2)</th>
<th>(x_3)</th>
<th>(x_4)</th>
<th>(s_1)</th>
<th>(s_2)</th>
<th>(s_3)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sales mix</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>-1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Hours components</td>
<td>-(\frac{1}{4})</td>
<td>0</td>
<td>0</td>
<td>(\frac{3}{4})</td>
<td>(\frac{1}{4})</td>
<td>-(\frac{1}{4})</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>(\frac{3}{4})</td>
<td>1</td>
<td>0</td>
<td>(\frac{1}{4})</td>
<td>(\frac{3}{4})</td>
<td>-(\frac{3}{4})</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>10</td>
<td>0</td>
<td>10</td>
<td>-10</td>
<td>0</td>
<td>30</td>
<td></td>
</tr>
</tbody>
</table>

\[ \left( \begin{array}{c} x_3 \\ x_2 \\ x_1 \end{array} \right) = \left( \begin{array}{c} 700 \\ 1525 \\ 1625 \end{array} \right) + \delta \left( \begin{array}{c} \frac{700}{3} \\ \frac{1525}{3} \\ \frac{1625}{3} \end{array} \right) \]

So we would be willing to buy up to \(\frac{6100}{3}\) component at any price up to \(30\). For prices above \(30\), profit would go down per unit so it would not be worth buying anything.

\[ \begin{array}{c|ccccc|c}
\hline
\text{a) Change RHS of 3rd constraint by} \ 
\delta \geq 0 \ 
\text{we get} \\
\left( \begin{array}{c} x_3 \\ x_2 \\ x_1 \end{array} \right) & = & \left( \begin{array}{c} 700 \\ 1525 \\ 1625 \end{array} \right) + \delta \left( \begin{array}{c} \frac{700}{3} \\ \frac{1525}{3} \\ \frac{1625}{3} \end{array} \right) \\
\end{array} \]

\(0 \leq \delta \leq \frac{1525}{3} = \frac{6100}{3}\)

\[ \text{so we need} \quad \delta \leq \frac{1525}{3} \]

\[\text{b) Change RHS of 2nd constraint by} \ 
\delta, \quad \left( \begin{array}{c} x_1 \\ s_2 \\ x_3 \end{array} \right) = \left( \begin{array}{c} 700 \\ 1525 \\ 1625 \end{array} \right) + \delta \left( \begin{array}{c} \frac{1}{3} \\ \frac{1}{3} \\ \frac{1}{3} \end{array} \right) \]

Note that we have a "s" constraint so we use the artificial column \(A_1\).

This implies \(-700 \leq \delta \leq \frac{1625 - \frac{y}{3}}{5} = 1300\). The objective of \(A_1\) is \(-10\) so the profit change is \(-10\delta\), thus if we drop the sales constraint entirely by setting \(\delta = -700\) our profit goes up by \(7000\).\$

\[\text{c) Change the coeff of} \ x_3 \ \text{in 2nd constraint from} \ y \ \text{to} \ \delta, \ \text{and keeping the same}\]

\(x\)-solution (i.e. \(x_1 = x_2 = 1625, x_3 = 700\)) will not change feasibility since the slack would be \(5 \times 1525 - 700 > 0\). However we cannot claim that the solution is optimal without either solving from scratch or using complementary slackness. We have dual variables \(y_1, y_2, y_3\) for 1st, 2nd and 3rd constraint.

1st and 2nd constraints - 2nd constraint is slack so \(y_2 = 0\). \(x_2\) and \(x_3\) are > 0 so we look at the 2nd and 3rd dual constraints.
Note: This has changed from 4 to 5.

\[ 3y_2 + 5y_3 \leq 120 \quad \text{dual constraint for } x_2 \]
\[ y_1 + 5y_2 + 5y_3 \geq 0 \quad 160 \quad \text{(-1, } x_3) \]

Which need to be tight. Note we already know \( y_2 = 0 \).

1st constraint in dual problem is \( = \) so \( y_1 \leq 0 \). Then in the dual

\[ y_2 = 20 \quad \text{and} \]
\[ y_1 + 5y_3 = 160 \quad \Rightarrow \quad y_1 = 40. \]

However we need \( y_2 \).

So, since the \( 4 \rightarrow 5 \) was the coefficient of \( y_2 \) which is 0 since
then was slack in the 2nd constraint this is the same dual problem
as the original. Therefore the solution is still optimal.

d) Currently we stand to lose 10$ for every unit of the \( y_2 \) line, thus the
price of the \( y_2 \) line needs to increase by 10$, to 170$ or higher for
it to be profitable. If the price is below 170, the current
basis remains optimal.

e) changing obj. coeff. of \( x_3 \) by \( \delta \), we need dual var. for non-basic var.
to be 0$, i.e.

\[
\begin{pmatrix} 10 & 10 & 4 & 10 & 30 \end{pmatrix} + \delta \begin{pmatrix} 0 & 1 & -1 & 0 \end{pmatrix} = \begin{pmatrix} 10 & 10+\delta & 10-\delta & 30 \end{pmatrix}
\]

\( x_1 \quad x_2 \quad s_1 \quad s_3 \quad \delta \) (coeff. of non-basic in the row where
\( x_3 \) is basic.)

So, \(-10 \leq \delta \leq 10 \). If the basis is opt. as long as the obj. coeff. of \( x_3 \)
\( \delta \) is in \([130, 150] \).

Note: We could have used \( \delta \) instead of \( s_1 \) and the dual price
would be \(-10 + \delta \) (the negative of \( s_1 \)) and we would require
it to be \( \leq 0 \) instead of \( \geq 0 \) as in the case of \( s_1 \), either way
you get the same bound on \( \delta \).