3.3.5

<table>
<thead>
<tr>
<th></th>
<th>X₁</th>
<th>X₂</th>
<th>X₃</th>
<th>S₁</th>
<th>S₂</th>
<th>S₃</th>
</tr>
</thead>
<tbody>
<tr>
<td>-2</td>
<td>1</td>
<td>4</td>
<td>1/2</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>3</td>
<td>0</td>
<td>3</td>
<td>-1/2</td>
<td>1</td>
<td>0</td>
<td>9</td>
</tr>
<tr>
<td>2</td>
<td>0</td>
<td>-1</td>
<td>-1</td>
<td>0</td>
<td>1</td>
<td>8</td>
</tr>
<tr>
<td>-4</td>
<td>0</td>
<td>-5</td>
<td>3</td>
<td>0</td>
<td>0</td>
<td>64</td>
</tr>
</tbody>
</table>

(a) The basic feasible solution corresponding to this tableau is:

\[ \begin{align*}
X₁ &= 0 \\
X₂ &= 24 \\
X₃ &= 0 \\
S₁ &= 0 \\
S₂ &= 9 \\
S₃ &= 8
\end{align*} \]

(b) We pivot \( x₁ \) into the basis, the replacement quantities are

\[ \begin{align*}
\frac{1}{6} &= 3 \\
\frac{1}{2} &= 4
\end{align*} \]

So we pivot on the second row, so \( S₂ \) leaves the basis. First divide through row 2 by 3 and get

\[ \begin{align*}
\frac{X₁}{6} &= \frac{1}{2} \cdot \frac{1}{3} \cdot 0 \\
\frac{X₂}{6} &= \frac{1}{2} \cdot \frac{1}{3} \cdot 9 \\
\frac{X₃}{6} &= \frac{1}{2} \cdot \frac{1}{3} \cdot 8 \\
\frac{S₁}{6} &= \frac{1}{2} \cdot \frac{1}{3} \cdot 0 \\
\frac{S₂}{6} &= \frac{1}{2} \cdot \frac{1}{3} \cdot 76
\end{align*} \]

(c) The simplex tableau obtained in (b) is not optimal since one entry in the objective row is negative.

3.3.6

Maximize \( 30X₁ + 40X₂ \)

Subject to

\[ \begin{align*}
2X₁ + 5X₂ &\leq 90 \\
3X₁ + 4X₂ &\leq 100 \\
X₁, X₂ &\geq 0
\end{align*} \]

We start with the initial tableau

<table>
<thead>
<tr>
<th></th>
<th>X₁</th>
<th>X₂</th>
<th>S₁</th>
<th>S₂</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>5</td>
<td>1</td>
<td>0</td>
<td>90</td>
</tr>
<tr>
<td>3</td>
<td>4</td>
<td>0</td>
<td>1</td>
<td>100</td>
</tr>
</tbody>
</table>
| -30| -40| 0  | 0  | 0 

We pivot \( X₁ \) into the basis, the replacement quantities are

\[ \frac{90}{2} = 45 \]

\[ \frac{100}{3} = \frac{100}{3} < 45 \]

So \( S₂ \) leaves the basis.

We pivot \( X₁ \) into the basis, the replacement quantities are

\[ \frac{90}{2} = 45 \]

\[ \frac{100}{3} = \frac{100}{3} < 45 \]

So \( S₂ \) leaves the basis.

See graph for visualization.
The next tableau is then

$$
\begin{array}{cccccc}
X_1 & X_2 & S_1 & S_2 & 1 & 30/3 \\
0 & 1/3 & 1 & -3/3 & 1 & 70/3 \\
1 & 4/3 & 0 & 1/3 & 1 & 10/3 \\
0 & 0 & 0 & 10 & 1000 \\
\end{array}
$$

This corresponds to one optimal basic feasible solution, since

$$
X_1 = 10/3 \\
X_2 = 0 \\
S_1 = 70/3 \\
S_2 = 0
$$

the entries in the obj. row are all non-negative.

Since $X_2$ has a 0 in the obj. row and is non-basic we get another optimal solution by pivoting $X_2$ into the basis. The replacement quantities are

$$
\begin{array}{c}
70/3 \\
9/3
\end{array}
= 10 \\
25
$$

So $S_1$ leaves the basis and the next tableau is

$$
\begin{array}{cccccc}
X_1 & X_2 & S_1 & S_2 & 1 & 10 \\
0 & 0 & 1/3 & -3/3 & 1 & 20 \\
0 & 0 & 0 & 1/3 & 1 & 20 \\
0 & 0 & 0 & 10 & 1000 \\
\end{array}
$$

which corresponds to

$$
X_1 = 20 \\
X_2 = 10 \\
S_1 = 20 \\
S_2 = 0
$$

And is also optimal.

3.3.11

$$
\begin{array}{cccccccc}
X_1 & X_2 & X_3 & S_1 & S_2 & S_3 & S_4 \\
5 & 1 & 1 & 1 & 0 & 0 & -3 & 40 \\
2 & 0 & 0 & 1 & 1 & 0 & 2 & 18 \\
1 & 2 & 0 & 0 & 1 & 3 & 12 \\
1 & 0 & 0 & 2 & 0 & 0 & 1 & 24 \\
\end{array}
$$

To simplify calculations we compute the replacement quantities for non-basic variables.

$$
X_1: \frac{40}{5} = 8 \quad \text{pivot on 1st row}
$$

$$
\begin{array}{c}
\frac{11}{2} = 9 \\
\frac{14}{3} = 9 \\
\frac{24}{5} = 24 \\
\end{array}
$$

$$
S_4: \frac{6}{36} = 36 \\
\phi = 36 \\
\frac{24}{36} = 36
$$

If we pivot $X_1$ into the basis by adding 1. Row 1 (since for $X_1$ we pivot in 1st row) to the obj. row and the obj. value would increase by 40 (the R.H.S value of Row 1) and become 285.

If we pivot $S_1$ into the basis we would add 4. Row 2 to obj. row (assuming we pivot on 2nd row) for an increase of $18 \times 4 = 72$ in obj. value.

If we pivot $S_4$ we would get 2.12 = 24 increase in obj. value.

For (c) we should pivot $X_1$ into the basis to get an obj. value of 285 and for (d) the greatest increase in obj. value comes from pivoting $S_1$ into the basis.
(b) The next solution is degenerate if a basic variable has value 0 in the basic feasible solution. This happens only if we have two (or more) minimum replacement quantities, i.e. if we pivot $s_1$.

(c) For $s_3$ to be 0 it would have to be non-basic (or degenerate). Since $s_3$ is in 3rd row we should pivot on $s_4$. For the degenerate case of pivoting $s_1$, only one of $s_1$ and $x_3$ will become non-basic (and both are 0).

(e) The objective function is given by

$$Z = 245 + 5x_1 + 25x_2 + 6s_4$$

with respect to this basis, so per unit of income for $x_1$ is 5, $s_1$ is 2 and $s_4$ is 6. Thus $s_4$ achieves the greater increase in the obj. value per unit.

3.3.14

max $30x_1 + 40x_2$

$$3x_1 + 4x_2 = 45$$

$$x_1 + x_2 = 14$$

$$x_1 = 12$$

$$x_2 = 20$$

Pivot $x_2$ into basis, the replacement quantities are $\frac{48}{12} = 4$ so we pivot on $s_1$.

Pivot $s_4$ out.

| $x_1$ $x_2$ $s_1$ $s_2$ $s_3$ |
|------------------|------------------|------------------|
| $\frac{3}{4}$ $1$ $1$ $0$ $0$ | $\frac{9}{4}$ | $\frac{1}{4}$ | $1$ | $0$ | $1\frac{1}{2}$ |
| $\frac{1}{4}$ $0$ $-\frac{1}{4}$ $1$ $0$ | $1\frac{1}{4}$ | $-\frac{1}{4}$ | $1$ | $0$ | $12$ |
| $0$ $0$ $1$ $0$ $0$ | $12$ | $0$ | $0$ | $0$ | $480$ |

Which is optimal but not unique, since the entries in the obj. row are all 20, but $x_1$ which is non-basic has value 0 in obj. row.

We get another optimal solution by pivoting $x_1$ into the basis. The replacement quantities are $\frac{12}{\frac{3}{4}} = 16$ so we pivot on 2nd row

$\frac{3}{4} = 8$ and get a new tableau.

$\frac{12}{1} = 12$ which is also optimal. The two tableaux correspond to the basic feasible solutions:

$$\begin{align*}
x_1 &= 0 \\
x_2 &= 12 \\
s_1 &= 0 \\
s_2 &= 2 \\
s_3 &= 12
\end{align*}$$

and

$$\begin{align*}
x_1 &= 8 \\
x_2 &= 6 \\
s_1 &= 0 \\
s_2 &= 0 \\
s_3 &= 4
\end{align*}$$

respectively.
3.2.4

3.3.2

From 3.2.4

\[
\begin{array}{cccccc}
X_1 & X_2 & S_1 & S_2 & S_3 \\
0 & \frac{1}{2} & 1 & 0 & -\frac{1}{6} & 110 \\
0 & \frac{1}{2} & 0 & 1 & -\frac{1}{6} & 60 \\
1 & \frac{1}{2} & 0 & 0 & \frac{1}{6} & 40 \\
0 & -7 & 0 & 0 & 34 & 400
\end{array}
\]

done

basic soln is

\[
X_1 = 40 \\
X_2 = 0 \\
S_1 = 110 \\
S_2 = 60 \\
S_3 = 0
\]

X_2 enters, \( c_{20} = \frac{220}{40} = 5.5 > \frac{20}{8} = 2.5 \)

optimal solution

\[
X_1 = 20, \ X_2 = 40
\]

opt value is 680
If we pivot \( x_2 \) into the basis, then there are no replacement quantities. This means we can increase \( x_2 \) as much as we want without violating any constraints. Since \( x_2 \) has a strictly negative value in the obj. row, this means we can increase the obj. value without any limits. Thus the linear program is unbounded.

Notice that any point \( x_2 = t \) satisfies the constraints for \( t \geq 0 \) and has obj. value \( t \), so we can get arbitrarily high obj. values.

Now \( -a_i = -40 - x_1 + x_2 - s_2 \) for each \( i \). The initial simplex tableau is then

\[
\begin{array}{cccccc}
X_1 & X_2 & S_1 & S_2 & a_i & \text{obj. coeff.} \\
\hline
1 & 0 & 1 & 0 & 0 & 10 \\
-1 & 1 & 0 & -1 & 1 & 40 \\
-4 & -8 & 0 & 0 & 0 & 0 \\
1 & -1 & 0 & 1 & 0 & -40 \\
\end{array}
\]

Since \( x_2 \) has negative artificial obj. row coefficient we pivot it into the basis. The replacement quantities are \( y_i = 10 \) so \( S_1 \) leaves the basis.

\[
\begin{array}{cccccc}
X_1 & X_2 & S_1 & S_2 & a_i & \text{obj. coeff.} \\
\hline
1 & 1 & 1 & 0 & 0 & 10 \\
-2 & 0 & -1 & -1 & 1 & 30 \\
4 & 0 & 1 & 1 & 0 & 40 \\
2 & 0 & 1 & 1 & 0 & -30 \\
\end{array}
\]

The artificial row has all entries \( \geq 0 \) so it is optimal with a value of \(-30\). This implies that the original problem is infeasible since if we had a feasible solution, the artificial variables would be \( 0 \), with an artificial obj. value of \( 0 \).
\[
\text{Max} \quad 4x_1 + 5x_2 + 3x_3 \\
x_1 + 2x_2 + 3x_3 \leq 40 \quad \text{add slack } s_1, \\
2x_1 + x_2 \geq 20 \quad \text{add surplus } s_2 \text{ and artificial } a_1, \\
x_1 - x_2 + 2x_3 = 15 \quad \text{add artificial } a_2 \\
x_1 \geq 0, x_2 \geq 0, x_3 \geq 0
\]

Now 
\[
2x_1 + 4x_2 + s_2 + a_1 = 20 \quad \text{and} \quad x_1 - x_2 + 2x_3 + a_2 = 15
\]

So isolating \( a_1 \) and \( s_2 \) we get
\[
-a_1 - s_2 = -(20 - 2x_1 - 4x_2 + s_2) - (15 - x_1 + x_2 - 2x_3)
\]
\[
= -35 + 3x_1 + 3x_2 + 2x_3 - s_2
\]

Which is the artificial objective. The initial simplex tableau is then

\[
\begin{array}{cccccccc}
X_1 & X_2 & X_3 & S_1 & S_2 & a_1 & a_2 \\
\hline
1 & 2 & 3 & 1 & 0 & 0 & 0 & 40 \\
2 & 4 & 0 & 0 & -1 & 1 & 0 & 20 \\
1 & -1 & 2 & 0 & 0 & 0 & 1 & 15 \\
-4 & -3 & -2 & 0 & 1 & 0 & 0 & 0 \\
-3 & -3 & -2 & 0 & 1 & 0 & 0 & -35
\end{array}
\]

\[3.6.8\]
\[
\text{Max} \quad x_1 + x_2 + 2x_3 + 4x_4
\]
\[
x_1 + 2x_2 + x_3 + x_4 \leq 400 \\
x_1 + 2x_2 + x_4 \geq 200 \\
x_2 + 2x_3 = 150
\]
\(x_1, x_2, x_3, x_4 \geq 0\)

and the initial tableau is

\[
\begin{array}{cccccccc}
X_1 & X_2 & X_3 & X_4 & S_1 & S_2 & a_1 & a_2 \\
\hline
1 & 2 & 1 & 1 & 0 & 0 & 0 & 400 \\
1 & 2 & 0 & 1 & 0 & 1 & 0 & 200 \\
0 & 1 & 2 & 0 & 0 & 0 & 1 & 150 \\
-1 & -1 & -2 & -4 & 0 & 0 & 0 & 0 \\
-1 & -3 & -2 & -1 & 0 & 0 & 0 & -350
\end{array}
\]

It pays off to start at replacement quantities to see if we can pivot an artificial variable out of the basis. This happens for all non-basic variables here. Pivot \( x_4 \) in

So \( a_1 \) leaves
Then the tableau becomes:

\[
\begin{array}{cccccccr}
X_1 & X_2 & X_3 & X_4 & S_1 & q_1 & q_2 \\
\hline
0 & 0 & 1 & 0 & 1 & -1 & 0 & 200 \\
1 & 2 & 0 & 1 & 0 & 1 & 0 & 200 \\
0 & 1 & 2 & 0 & 0 & 0 & 0 & 150 \\
\hline
3 & 5 & -2 & 0 & 0 & 4 & 0 & 800 \\
0 & -1 & -2 & 0 & 0 & 1 & 0 & -150 \\
\hline
\end{array}
\]

Now pivot \( X_3 \) into basis.

\(-\frac{1}{2}q_3\)

(new pivot \( X_3 \) would not get rid of an artificial variable). The replacement quantities are

\[
\begin{align*}
\frac{200}{1} & = 200 \\
\frac{150}{2} & = 75
\end{align*}
\]

so \( q_2 \) leaves.

Next, the artificial was on both non-basic and \( 0 \) we throw them away and get a tableau with a basic feasible solution.

\[
\begin{array}{cccccccr}
X_1 & X_2 & X_3 & X_4 & S_1 & q_1 & q_2 \\
\hline
0 & -\frac{1}{2} & 0 & 0 & 1 & -1 & -\frac{1}{2} & 125 \\
1 & 2 & 0 & 1 & 0 & 1 & 0 & 200 \\
0 & \frac{1}{2} & 1 & 0 & 0 & 0 & \frac{1}{2} & 75 \\
\hline
3 & 6 & 0 & 0 & 0 & 4 & 1 & 950 \\
0 & 0 & 0 & 0 & 1 & 1 & 0 \\
\hline
\end{array}
\]

Since obj row entries are \( 20 \) this is an optimal basic feasible solution.

\[
\begin{align*}
X_1 &= 0 \\
x_2 &= 0 \\
x_3 &= 75 \\
x_4 &= 200 \\
S_1 &= 125
\end{align*}
\]