Stability, categoricity and axiomatization of abstract elementary classes

Thesis defense

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Overview

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2. Abstract elementary classes (AECs)

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2. Abstract elementary classes (AECs)
3. Hanf number of the first stability cardinal in AECs
4. Axiomatizing AECs and applications
5. Stability results assuming tameness, monster model and continuity of nonsplitting
6. Categoricity transfer for tame AECs with amalgamation over sets

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  - Raise open questions

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Definition

Let L be a finitary language. An abstract elementary class $\mathbf{K} = \langle K, \leq_{\mathbf{K}} \rangle$ in $L = \mathrm{L}(\mathbf{K})$ satisfies the following axioms:

1. K is a class of L-structures and $\leq_{\mathbf{K}}$ is a partial order on K.
2. For $M_1, M_2 \in K$, $M_1 \leq_{\mathbf{K}} M_2$ implies $M_1 \subseteq M_2$ (as L-substructure).

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   a. If $M \in K$, N is an L-structure, $M \cong N$, then $N \in K$.
   b. Let $M_1, M_2, N_1, N_2 \in K$. If $f : M_1 \cong M_2$, $g : N_1 \cong N_2$, $g \supseteq f$ and $M_1 \leq_{\mathbf{K}} N_1$, then $M_2 \leq_{\mathbf{K}} N_2$.

$$\begin{tikzcd} N_1 \arrow[r, "g"] & N_2 \\ M_1 \arrow[u, "\leq_{\mathbf{K}}"] \arrow[r, "f"] & M_2 \arrow[u, dotted, "\leq_{\mathbf{K}}"'] \end{tikzcd}$$

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Definition (Continued)

4. Coherence: Let $M_1, M_2, M_3 \in K$. If $M_1 \leq_{\mathbf{K}} M_3$, $M_2 \leq_{\mathbf{K}} M_3$ and $M_1 \subseteq M_2$, then $M_1 \leq_{\mathbf{K}} M_2$.

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5. Löwenheim-Skolem axiom: There exists an infinite cardinal $\lambda \geq |\mathrm{L}(\mathbf{K})|$ such that: for any $M \in K$, $A \subseteq |M|$, there is some $N \in K$ with $A \subseteq |N|$, $N \leq_{\mathbf{K}} M$ and $\|N\| \leq \lambda + |A|$. We call the minimum such λ the Löwenheim-Skolem number $\mathrm{LS}(\mathbf{K})$.

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6. Chain axioms: Let α be an ordinal and $\langle M_i : i < \alpha \rangle \subseteq K$ such that for $i < j < \alpha$, $M_i \leq_{\mathbf{K}} M_j$.
    1. Then $M = \bigcup_{i<\alpha} M_i$ is in K and for all $i < \alpha$, $M_i \leq_{\mathbf{K}} M$.
    2. Let $N \in K$. If in addition for all $i < \alpha$, $M_i \leq_{\mathbf{K}} N$, then $M \leq_{\mathbf{K}} N$.

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$\mathbf{K}$ has the *amalgamation property* (AP) if for any $M_0, M_1, M_2 \in K$ with $M_0 \leq_{\mathbf{K}} M_1$ and $M_0 \leq_{\mathbf{K}} M_2$, then there exist $M_3 \in K$ and $f : M_1 \underset{M_0}{\longrightarrow} M_3$ such that $M_2 \leq_{\mathbf{K}} M_3$.

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Definition (Galois types)

Let $a_i \in N_i$ and $M_i \leq_{\mathbf{K}} N_i$ for $i = 1, 2$. We define
$(a_1, M_1, N_1) \sim (a_2, M_2, N_2)$ when $M_1 = M_2$ and there are $N \in K$,
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- Let κ be a cardinal. $\mathbf{K}$ is *κ-tame* if for any Galois types $p \neq q$ both in $\mathrm{gS}(M)$, there is $M_0 \leq M$, $\|M_0\| \leq \kappa$ such that $p \restriction M_0 \neq q \restriction M_0$.

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- Let κ be a cardinal. $\mathbf{K}$ is *κ-short* if for any Galois types $p \neq q \in \mathrm{gS}(M)$ of the same length, there is $|I| \leq \kappa$ such that $p^I \neq q^I$.

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Can we lower the bound of (2) to $2^{\mathrm{LS}(\mathbf{K})}$? Or are there counterexamples?

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Let's look at how (2) was proved!

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- Let μ be a cardinal. $\mathbf{K}$ has the *order property of length* μ (OP_μ) if there exist $\langle a_i : i < \mu \rangle$ (well-ordered!), $M \leq_{\mathbf{K}} N$ such that for $i_0 < i_1$ and $j_0 < j_1$, we have $\mathrm{gtp}(a_{i_0} a_{i_1} / M; N) \neq \mathrm{gtp}(a_{j_1} a_{j_0} / M; N)$.

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We filled in the details of (2) and (3) (Proposition 3.4, Theorem 6.1).

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Theorem (Proposition 4.1)

Let λ be an infinite cardinal and α be an ordinal with $\lambda \leq \alpha < (2^\lambda)^+$. Then there is a stable AEC $\mathbf{K}$ such that $\mathrm{LS}(\mathbf{K}) = \lambda$, $\mathbf{K}$ has the order property of length up to $\beth_\alpha(\lambda)$ and is unstable anywhere below $\beth_\alpha(\lambda)$. Moreover, $\mathbf{K}$ has JEP, NMM and $(< \aleph_0)$-tameness but not AP.*

What about a counterexample (with high instability and long order property)?

Theorem (Proposition 4.1)

Let λ be an infinite cardinal and α be an ordinal with $\lambda \leq \alpha < (2^\lambda)^+$. Then there is a stable AEC $\mathbf{K}$ *such that* $\mathrm{LS}(\mathbf{K}) = \lambda$, $\mathbf{K}$ *has the order property* of length up to $\beth_\alpha(\lambda)$ and is unstable anywhere below $\beth_\alpha(\lambda)$. Moreover,* $\mathbf{K}$ *has JEP, NMM and $(< \aleph_0)$-tameness but not AP.*

First stability cardinal/*OP* length	Tame+*AP*	Tame+$(\neg AP)$
Upper bound	$\beth_{(2^{\mathrm{LS}(\mathbf{K})})^+}$	?
Can go up to	?	$\beth_{(2^{\mathrm{LS}(\mathbf{K})})^+}$

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Other results:

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Other results:

- Our example is an $EC(\lambda, 2^\lambda)$ ordered by $\mathrm{L}(\mathbf{K})$-substructure;
- Defined OP^* in place of *OP*:
  - The index set of a_i can be linearly ordered;
  - Fact (Shelah+Vasey) still goes through.

Axiomatizing AECs and applications

We have the classical presentation theorem for AECs:

Fact (Shelah)

Let $\mathbf{K}$ be an AEC and $\lambda = \mathrm{LS}(\mathbf{K})$. Then K is $PC_{\lambda,2^\lambda}$.

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1. $K = PC(T, \Gamma, \mathrm{L}(\mathbf{K}))$;
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2. $|T| \leq \lambda$;
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Can we have a precise control of the parameter 2^λ in the statement?

We refined the proof of the classical presentation theorem and obtained:

Theorem (Theorem 4.1)

Let $\mathbf{K}$ *be an AEC and* $\lambda = \mathrm{LS}(\mathbf{K})$. *Then there is* χ *depending on* $\mathbf{K}$ *such that* $\lambda \leq \chi \leq 2^{\lambda}$ *and* K *is* $PC_{\chi,\chi}$.

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What is this χ? $\chi = \lambda + I_2(\lambda, \mathbf{K})$.

Definition

- $I(\lambda, \mathbf{K}) = |\{M/_{\cong} : M \in K_\lambda\}|$

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- $I(\lambda, \mathbf{K}) = |\{M/_{\cong} : M \in K_{\lambda}\}|$
- $I_2(\lambda, \mathbf{K}) = |\{(M, N)/_{\cong} : M \leq_{\mathbf{K}} N \text{ both in } K_{\lambda}\}|$ where $(M_1, N_1) \cong (M_2, N_2)$ iff $M_1 \leq_{\mathbf{K}} N_1$, $M_2 \leq_{\mathbf{K}} N_2$ and there is $g : N_1 \cong N_2$ such that $g \restriction M_1 : M_1 \cong M_2$.

In general, computing I_2 is difficult. We can bypass such computation with more assumptions:

Theorem (Corollary 4.10)

Under $2^\lambda < 2^{\lambda^+}$, if $\mathbf{K}$ is categorical in λ, λ^+ and stable in λ, then $\mathbf{K}$ is $PC_{\lambda,\lambda}$.

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- Either
  - (4.1) directly encode the ordering (isomorphism types of pairs); or
  - (4.10) use coherence and uniqueness of limit models of the same cofinality.

Our result allows us to remove one of the assumptions in Shelah's theorem:

Fact (Shelah)

Let $\mathbf{K}$ be an AEC, $\theta \geq \mathrm{LS}(\mathbf{K})$. Suppose the following hold:

1. $K, K^{<}$ are both $PC_{\theta,\theta}$;
2. $\mathbf{K}$ is categorical in both θ and θ^{+};
3. $\delta(\theta, 1) = \theta^{+}$. (Threshold cardinal for an infinite decreasing chain to exist in a $PC_{\theta,1}$-class.)

Then $K_{\theta^{++}} \neq \emptyset$.

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Then $K_{\theta^{++}} \neq \emptyset$.

By 4.1, (1) is true for $\theta \geq \chi$.
By 4.10, under $2^{\mathrm{LS}(\mathbf{K})} < 2^{\mathrm{LS}(\mathbf{K})^+}$ and stability in $\mathrm{LS}(\mathbf{K})$, (2) already implies (1) for $\theta = \mathrm{LS}(\mathbf{K})$.

Fact (Shelah-Villaveces)

Let $\mathbf{K}$ be an AEC and $\lambda = \mathrm{LS}(\mathbf{K})$. Then K is axiomatizable (in $\mathrm{L}(\mathbf{K})$) in L_{θ,λ^+} where $\theta = (2^{2^{\lambda^+}})^{+++}$.

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Our proof strategy allows us to show:

Theorem (3.7)

Let $\mathbf{K}$ be an AEC, $L = \mathrm{L}(\mathbf{K})$, $\lambda = \mathrm{LS}(\mathbf{K})$ and $\chi = \lambda + I_2(\lambda, \mathbf{K})$. Then $\mathbf{K}$ can be axiomatized by a sentence in $L_{\chi^+,\lambda^+}(\omega \cdot \omega)$ (game quantification of $\omega \cdot \omega$ steps).

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The previous variation applies too (Corollary 3.14)!

We can also generalize our results to μ-AECs (which have weaker closure properties and whose language can be infinitary):

K is	**K** is axiomatizable in	**K** is
An AEC	$L_{\chi^+,\lambda^+}(\omega \cdot \omega)$	$PC_{\chi,\chi}$
A μ-AEC	$L_{\chi^+,\lambda^+}(\mu \cdot \mu)$	$PC^{\mu}_{\chi,\chi}$

Stability results assuming tameness, monster model and continuity of nonsplitting

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- Stability in a tail (from $2^{|T|}$ onwards);
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- Stability and $\kappa(T) = \aleph_0$ (finite character of forking);
- Stability and boundedness of the D-rank;
- Tree property...

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Many generalizations to AECs had been obtained with usually high thresholds (the first Hanf number) and cardinal jumps.

In addition to tameness and monster model, we add the assumption of *continuity of nonsplitting*, we were able to improve the thresholds, reduce the cardinal jumps (mostly a successor) and expand the list of equivalent criteria (Theorem 8.2):

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1. **K** has $\aleph_0$-local character of μ-nonsplitting;
2. There is a good frame over the limit models in K_μ ordered by $\leq_u$, except for symmetry. In this case the frame is canonical;
3. K_μ has uniqueness of limit models;
4. For any increasing chain of μ^+-saturated models, the union of the chain is also μ^+-saturated;
5. K_{μ^+} has a superlimit;
6. **K** is (μ^+, μ^+)-solvable;
7. **K** is stable in $\geq \mu$ and has continuity of $\mu^{+\omega}$-nonsplitting;
8. *U*-rank is bounded when μ-nonforking is restricted to the limit models in K_μ ordered by $\leq_u$.

By "localizing" superstability, we managed to obtain another list for (strictly) stable theories:

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1. **K** has δ-local character of μ-nonsplitting;
2. There is a good frame over the skeleton of $(\mu, \geq \delta)$-limit models ordered by $\leq_u$, except for symmetry and local character δ in place of $\aleph$. In this case the frame is canonical;
3. **K** has uniqueness of $(\mu, \geq \delta)$-limit models;
4. For any increasing chain of μ^+-saturated models, if the length of the chain has cofinality $\geq \delta$, then the union is also μ^+-saturated;
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The criteria in the list are equivalent *modulo extra stability* (see slides 33-34).

Categoricity transfer for tame AECs with amalgamation over sets

Fact (Morley, Shelah)

Let T be a first-order theory. If T is categorical in some cardinal $> |T|$, then it is categorical in all cardinals $> |T|$.

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A central test question for classification theory for non-elementary classes is the following:

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A central test question for classification theory for non-elementary classes is the following:

Conjecture (Shelah)

Let $\mathbf{K}$ be an AEC and $\lambda = \mathrm{LS}(\mathbf{K})$. The threshold for categoricity transfer is $\beth_{(2^\lambda)^+}$. Namely, if $\mathbf{K}$ is categorical in some $\mu \geq \beth_{(2^\lambda)^+}$, then it is categorical in all $\mu \geq \beth_{(2^\lambda)^+}$.

Approaches towards the conjecture include:

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We follow (1) and focus on the *upward* categoricity transfer:

Question

If $\mathsf{LS}(\mathbf{K}) = \lambda$, can we find a cardinal μ_λ such that (categoricity in some $\mu \geq \mu_\lambda$) $\rightarrow$ (categoricity in all $\mu' \geq \mu$)?

We call μ_λ the *threshold* (ideally $\mu_\lambda = \lambda^+$.)

Fact (Grossberg-VanDieren)

Let **K** be an LS(**K**)-tame AEC with a monster model. If **K** is categorical in some *successor* $\mu > \mathrm{LS}(\mathbf{K})$, then it is categorical in all $\mu' > \mu$.

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Fact (Vasey)

1. Let $\mathbf{K}$ be an $\mathrm{LS}(\mathbf{K})$-tame AEC with a monster model and have *primes*. If $\mathbf{K}$ is categorical in some (not necessarily successor) $\mu > \mathrm{LS}(\mathbf{K})$, then it is categorical in all $\mu' > \mu$.

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- Can we remove the assumption of primes in (1)?

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2. In elementary classes, for any stability cardinal $\lambda \geq |T|$, λ-saturated models have primes.

Question

- Can we remove the assumption of primes in (1)?
- The proof of (2) is syntactic (using results of Shelah). Is there a semantic criterion to obtain primes?

It would be ideal to have "tameness + monster model $\to$ threshold is $\mathrm{LS}(\mathbf{K})^+$". An approximation is the following:

Fact (Vasey)

Shortness + monster model $\to$ primes for *saturated models*

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This fact does not help much with the categoricity conjecture, because the class of saturated models (instead of λ-saturated models for a fixed λ) is always totally categorical.

Later Shelah-Vasey investigated the notion of *excellence* for AECs and obtained sufficient conditions for excellence using *non-ZFC axioms*.

Fact

1. (Shelah-Vasey) Excellent classes imply tameness, monster model and primes for $(\mathrm{LS}(\mathbf{K})^+)$-saturated models (hence the threshold is $\mathrm{LS}(\mathbf{K})^+$);

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Can we replace WGCH by model theoretic properties?
Yes, if we assume *amalgamation over sets* in place of amalgamation (shortness will follow from tameness).

Definition

$\mathbf{K}$ has the *amalgamation property over set bases* (*AP over sets*) if for any $M_1, M_2 \in K$, any $A \subseteq |M_1| \cap |M_2|$, there is $M_3 \in K$ and $f : M_1 \underset{A}{\rightarrow} M_3$ such that $M_2 \leq_{\mathbf{K}} M_3$.

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Fact

Complete first-order theories have amalgamation over sets if we work in a monster model.

Theorem (6.13)

Let $\mathbf{K}$ *be an AEC which is* $\mathrm{LS}(\mathbf{K})$*-tame and has a monster model with amalgamation over sets. Suppose* $\mathbf{K}$ *is categorical in some* $\xi > \mathrm{LS}(\mathbf{K})$*, then it is categorical in all* $\xi' \geq \min(\xi, h(\mathrm{LS}(\mathbf{K})))$*.*

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Upper bound	$\beth_{(2^{\mathrm{LS}(\mathbf{K})})^+}$	?
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Thank you for listening!

Auxiliary definitions

1. $M = \bigcup_{i<\delta} M_i$ is a limit model if $\langle M_i : i < \delta \rangle$ is increasing and continuous, and M_{i+1} is universal over M_i for each $i < \delta$.
2. Splitting: $p \in \mathrm{gS}(N)$ splits over M if there exists $f : M_1 \cong_M M_2$ such that $M \leq M_i \leq N$ and $f(p) \restriction M_2 \neq p \restriction M_2$.
3. Continuity of nonsplitting: if $\langle p_i : i \leq \delta \rangle$ is an increasing and continuous chain of types such that p_i does not split over M for $i < \delta$, then p_δ also does not split over M.
4. $\lambda'(\mathbf{K})$ is the least stability cardinal such that the local character of nonforking stabilizes (under continuity of nonsplitting, the local character is a decreasing function).

Definition

Let λ be an infinite cardinal and $\kappa \geq 1$. $\delta(\lambda, \kappa)$ is the minimum ordinal δ such that:

- For any first-order language L that contains a binary relation $<$ and a unary predicate Q,
- any first-order theory T in L of size $\leq \lambda$,
- any set of T-types Γ of size $\leq \kappa$,
- if there exists $M \in EC(T, \Gamma)$ with $(Q^M, <^M)$ of order type $\geq \delta$,
- then there is $N \in EC(T, \Gamma)$ with $(Q^N, <^N)$ ill-founded.

Further on Slide 23

Why the extra stability assumption?
The key direction on the list of superstable criteria is (1)$\Rightarrow$(3):

- Stability implies NOP;
- NOP + enough stability imply symmetry (for nonsplitting);
- Symmetry implies uniqueness of limit models.

In the second step, we require stability to "catch up" the order property, but there is no precise bound for the order property length (or counterexample with AP)!

Would extra stability already implies superstability, hence our result is trivial? A short answer is we do not know. The superstable criteria (7)$\Rightarrow$(1) says the following:

Theorem (Proposition 7.5)

There is $\lambda < h(\mu^{+\omega})$ such that if $\mathbf{K}$ is stable in $[\mu, \lambda)$ and has continuity of $\mu^{+\omega}$-nonsplitting, then it is $\mu^{+\omega}$-superstable.

The proof again makes uses of the order property, but this time in $\mu^{+\omega}$ instead of μ, so the required length λ could be longer.
With a different approach, Vasey obtained the following:

Fact (Vasey)

$\lambda'(\mathbf{K}) < h(\mathrm{LS}(\mathbf{K}))$: namely if $\mathbf{K}$ is superstable, the starting cardinal of stability-in-a-tail is bounded above by the first Hanf number.

A deeper understanding of the order property, symmetry and λ' would be significant.