Coulomb friction and oscillation: stabilization in finite time for a system of damped oscillators.

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Abstract

In this communication we made a first presentation of a set of results on the stabilization in a finite time of some mechanical processes where a Coulomb friction (or solid friction) term coexists with other physical frameworks leading to oscillations in absence of friction. Here, in particular, we concentrate our attention in some finite-dimensional dynamical systems which can be obtained by the modelling of N-coupled oscillators as well as trough a spatial discretization of a vibrating string equation in presence of a solid friction.

Introduction

The purpose of this work is to made a first presentation of the study made by the authors on the dynamics of the finite-dimensional system corresponding to vibration of N-particles of equal mass m located along the interval $(0,1)$ of the x axis. Each particle is connected to its neighbors by two harmonic springs of strength k, the elongation of the left one is given by $x_i(t)$ and we assume the motion subject to a resultant friction force which is the composition of a Coulomb (or solid) friction and other type of frictions such as, for instance, the one due to the viscosity of an surrounding fluid. The equations of motion for this system are

$$(P_N)\left\{\begin{array}{l} m\ddot{x}_i(t)+k(-x_{i-1}(t)+2x_i(t)-x_{i+1}(t))+\mu_\beta\beta(\dot{x}_i(t))+\mu_g g(\dot{x}_i(t))\ni 0 \\ x_i(0)=u_{0,i},\ \dot{x}_i(0)=v_{0,i} \end{array}\right.$$

$i=1,\ldots,N$, where we are assuming that $x_0(t)=0$, $x_{N+1}(t)=0$ for any $t\in(0,+\infty)$, μ_β, μ_g are positive constants, the term $\mu_\beta\beta(\dot{x}_i(t))$ represents the Coulomb friction, with β given by the maximal monotone graph in $\mathbb{R}^2$

$$\beta(r)=\{-1\} \text{ if } r<0, \beta(0)=[-1,1],\ \beta(r)=\{1\} \text{ if } r>0,$$

g is a Lipschitz continuous function such that $g(0)=0$, $\mu_\beta\beta(r)+\mu_g g(r)>0$ for all $r>0$ and the reverse inequality for $r<0$. The *internal* initial data $(u_{0,i})$, $(v_{0,i})$ are given in $\mathbb{R}^N$.

It is well known that, if we write, for simplicity, $k=\frac{1}{h^2}$ (with $h=1/(N+1)$) and $m=1$, then problem (P_N) arises in the spatial discretization, by finite differences, of the damped string equation

$$(P_\infty)\begin{cases} u_{tt}-u_{xx}+\mu_\beta\beta(u_t)+\mu_g g(u_t)\ni 0 & \text{in } (0,1)\times(0,+\infty),\\ u(0,t)=u(1,t)=0, & t\in(0,+\infty),\\ u(x,0)=u_0(x), u_t(x,0)=v_0(x) & x\in(0,1).\end{cases}$$

In fact, it was by passing to the limit, $N\to\infty$ in (P_N), how the wave equation (without friction) was obtained by Jean Le Rond D'Alembert in 1746.

The main goal of this paper is to give several criteria in order to have the stabilization in a finite time for this mechanical system. The study of the special case of a single oscillator, $N=1$, without viscous friction,

$$m\ddot{x}+2kx+\mu_\beta\beta(\dot{x})\ni 0,$$

can be found in many textbooks (see, for instance, [17]). It is easy to see then that the motion stops definitively after a finite time: i.e. there exists $T_e<+\infty$ and $x_\infty\in[-\frac{\mu_\beta}{2k},\frac{\mu_\beta}{2k}]$ such that $x(t)\equiv x_\infty$ for any $t\geq T_e$. There are, also, some partial results on the stabilization to an equilibrium state in a finite time for the solutions of the wave equation (see [7] and [8] for some particular initial data). The case of arbitrary initial data $u_0(x)$ and $v_0(x)$ seems to be, still, an open problem.

Concerning the case of $N-$particles we can mention the work by Bamberger and Cabannes [3] in which they prove the stabilization in a finite time in absence of viscous friction ($\mu_g=0$). We point out that this type of friction arises very often in the applications and that its consideration was already proposed by Lord Rayleigh (see, e.g. [18]). Concrete expressions for g can be found also in [17]. The case of a linear damping $g(\dot{x}_i)=\lambda\dot{x}_i$ and the absence of stabilization in a finite time for λ large enough was commented at the end of the paper [3] but no mention to the possibility of a simultaneous dichotomy of behaviors was made there.

One of our main goals is to prove that the presence of a viscous friction may originate a qualitative distinction among the orbits in the sense that the state of the system $\mathbf{x}(t):=(x_1(t),x_2(t),...,x_N(t))^T$ (here $\mathbf{h}^T$ means, in general, the trasposed vector of $\mathbf{h}$) may reach an equilibrium state in a finite time or merely in an asymptotic way (as $t\to+\infty$), according the initial data $\mathbf{x}(0)=\mathbf{x}_0:=(u_{0,1},u_{0,2},...,u_{0,N})^T$ and $\dot{\mathbf{x}}(0)=\mathbf{v}_0:=(v_{0,1},v_{0,2},...,v_{0,N})^T$. This dichotomy seems to be new in the literature and contrasts with the phenomena of *finite extinction time* for first order (in time) ordinary and parabolic nonlinear equations (see, for instance, the exposition made in [2]). Some results exhibiting this alternative, but for the case of a single particle with a non-Lipschitz friction term $\beta(u)=|u|^{\alpha-1}u$ ($\alpha\in(0,1)$), can be found in [11], [12] and [1] (problem raised, many year ago, by Haïm Brezis). We end the paper by showing that this alternative may occur also in the case of the wave equation (P_∞) in all dimension in space and under suitable conditions.

On the dichotomy for the N-dimensional system

The system under study can be written, in short, as a vectorial problem

$$(\mathbf{P}_N)\begin{cases} m\ddot{\mathbf{x}}(t)+k\mathbf{A}\mathbf{x}(t)+\mu_\beta\mathbf{B}(\dot{\mathbf{x}}(t))+\mu_\beta\mathbf{G}(\dot{\mathbf{x}}(t))\ni\mathbf{0},\\ \mathbf{x}(0)=\mathbf{x}_0,\ \dot{\mathbf{x}}(0)=\mathbf{v}_0\end{cases}$$

where $\mathbf{x}(t) := (x_1(t), x_2(t), ..., x_N(t))^T$, $\mathbf{A}$ is the symmetric positive definite matrix of $\mathbb{R}^{N\times N}$ given by

$$\mathbf{A} = \begin{pmatrix} 2 & -1 & 0 & ... & 0 \\ -1 & 2 & -1 & 0 & ... \\ 0 & -1 & 2 & -1 & 0 \\ ... & 0 & -1 & 2 & -1 \\ 0 & ... & 0 & -1 & 2 \end{pmatrix},$$

$\mathbf{B} : \mathbb{R}^N \to \mathcal{P}(\mathbb{R}^N)$ denotes the (multivalued) maximal monotone operator given by $\mathbf{B}(y_1, \ldots, y_N) = (\beta(y_1), \ldots, \beta(y_N))^T$ and $\mathbf{G} : \mathbb{R}^N \to \mathbb{R}^N$ is the Lipschitz continuous function defined by $\mathbf{G}(y_1, \ldots, y_N) = (g(y_1), \ldots, g(y_N))^T$. In what follows, $\mathbf{a} \cdot \mathbf{b}$ denotes the Euclidian scalar product of $\mathbf{a}, \mathbf{b} \in \mathbb{R}^N$ and $\| \, . \, \|$ the Euclidean norm.

Our first result deals with the existence, uniqueness and asymptotic behavior of solutions of $(\mathbf{P}_N)$

Theorem 1. *For any initial datum $(\mathbf{x}_0, \mathbf{v}_0) \in \mathbb{R}^{2N}$, the Cauchy problem $(\mathbf{P}_N)$ admits a unique weak solution $\mathbf{x} \in C^1([0, +\infty) : \mathbb{R}^N)$. Moreover, there exists a unique equilibrium state $\mathbf{x}_\infty \in \mathbb{R}^N$, i.e. satisfying that $\mathbf{A}\mathbf{x}_\infty (\in [-\frac{\mu_\beta}{2k}, \frac{\mu_\beta}{2k}]^N)^T$, such that*

$$\| \dot{\mathbf{x}}(t) \| + \| \mathbf{x}(t) - \mathbf{x}_\infty \| \to 0 \quad \text{as } t \to +\infty. \tag{1}$$

Concerning the dichotomy mentioned at the introduction, the following result shows that the stabilization in a finite time depends of the structural behavior of the viscous friction g near 0.

Theorem 2. *i) Suppose that $g(r)r \leq 0$ in some neighborhood of 0. Then all solutions of $(\mathbf{P}_N)$ stabilize in a finite time.*

ii) Suppose that $g(r) = \lambda r$ with $\lambda \geq \frac{2\sqrt{\lambda_1 mk}}{\mu_g}$, where λ_1 denotes the first eigenvalue of $\mathbf{A}$. Then there exist solutions of $(\mathbf{P}_N)$ which do not stabilize in any finite time.

iii) Suppose that $N = 1$, $A = 1 \in \mathbb{R}$ and g is C^1 in some neighborhood of 0. Then, if $g'(0) < \frac{2\sqrt{mk}}{\mu_g}$, all solutions stabilize in finite time but if $g'(0) \geq \frac{2\sqrt{mk}}{\mu_g}$ there exist some solutions which do not stabilize in any finite time.

Remark 1. Notice that the growth condition on $g(r)$, near $r = 0$, is independent on μ_β. In the case of a single particle (notice that then $\lambda_1 = 1$) more precise results can be obtained by using, as in [11], [12], [1], the trajectory equation in the phase space $y_x \in \frac{-kx - \mu_\beta \beta(y) - \mu_g g(y)}{y}$ but they will not presented here.

Remark 2. The positive results on stabilization in a finite time remain true for a general symmetric and positive definite matrix $\mathbf{A}$ as well as under the presence of some *impulsive forces* $\mathbf{f}(t)$ leading to the system

$$m\ddot{\mathbf{x}}(t) + k\mathbf{A}\mathbf{x}(t) + \mu_\beta \mathbf{B}(\dot{\mathbf{x}}(t)) + \mu_\beta \mathbf{G}(\dot{\mathbf{x}}(t)) \ni \mathbf{f}(t)$$

assuming that their amplitude is small enough: more precisely if

$$\exists \alpha > 0 \text{ such that } \mu_\beta \beta(r) + \mu_g g(r) \geq \alpha \text{ and } g(-r) = g(r) \text{ for any } r > 0$$

then we have to we assume that

$$\mathbf{f}(t) \in ([-\alpha + \epsilon, \alpha - \epsilon]^N)^T, \text{ for some } \epsilon \in [0, \alpha) \text{ and for a.e. } t \geq T_f, \text{ for some } T_f \geq 0.$$

This behavior face up to with the case in which the amplitude of $\mathbf{f}(t)$ becomes large and $g'(v) < 0$ for any $v \neq 0$. Then, the dynamics generates a wide range of events leading to the chaos (see [9]).

Remark 3. The simultaneous possibility of the occurrence of stabilization in a finite or infinite time does not hold for solutions of scalar first order in time equations of the form

$$u_t - d\Delta u + \beta(u) \ni 0 \tag{2}$$

for $\beta(u)$ multivalued at $u = 0$ and $d \geq 0$ (see, for instance, [6], [10] and their references). We assume given homogeneous Dirichlet boundary conditions and an initial datum. Moreover, if we add an extra term, $g(u)$, such that, $g(u)u \geq 0$ for any $u \in \mathbb{R}$, then the solutions of

$$U_t - d\Delta U + \beta(U) + g(U) \ni 0 \tag{3}$$

satisfy that $\|u(t,.)\|_{L^p(\Omega)} \geq \|U(t,.)\|_{L^p(\Omega)}$ and so, the extinction in a finite time of $u(t,.)$ implies the same property for $U(t,.)$. The opposite comparison holds when $g(u)u \leq 0$. This explain the important different behaviors among the solutions of problems of first and second order in time. Notice that if we assume $k = 0$ in $(\mathbf{P}_1)$ then we get that $U(t) = \dot{x}(t)$ satisfies an equation similar to 3 with $d = 0$. Notice, also, that if m is very small then problem (P_1) becomes a *quasi-static problem* (in the terminology of [13]) and then the solutions are closed to the solutions of the first order in time problem

$$(QSP_1) \begin{cases} 2kx + \mu_\beta \beta(\dot{x}) + \mu_g g(\dot{x}(t)) \ni 0, \\ x(0) = x_0 \end{cases}$$

In that case, $g(u)u \geq 0$ implies an opposite comparison to the above mentioned one with respect the solutions with $g = 0$. Nevertheless, the multivalued character of β at $u = 0$ does not imply, now, the stabilization in a finite time for the solutions of (QSP_1).

Proof of Theorem 1. To reformulate $(\mathbf{P}_N)$ in the framework of nonlinear semi-group operators theory we introduce the *phase space* $\mathbf{H} = (\mathbb{R}^N, <,>_\mathbf{A}) \times (\mathbb{R}^N, \cdot)$, with $<\mathbf{a}, \mathbf{b}>_\mathbf{A} = \mathbf{Aa} \cdot \mathbf{b}$, and we define the operator $\mathbf{L}$ in $\mathbf{H}$ by

$$\mathbf{L}(\mathbf{x}, \mathbf{y}) = \{-\mathbf{y}\} \times \{\frac{k}{m}\mathbf{Ax} + \frac{\mu_\beta}{m}\mathbf{B}(\mathbf{y})\} \text{ for } (\mathbf{x}, \mathbf{y}) \in \mathbf{H}. \tag{4}$$

It is easy to prove that $\mathbf{L}$ is maximal monotone in H and since $\frac{\mu_g}{m}\mathbf{G}(\mathbf{y})$ is Lipschitz continuous, by using the results on Lipschitz perturbations of maximal monotone operators (see [5]) we get the existence and uniqueness of a solution of $(\mathbf{P}_N)$. Multiplying the equation by $\dot{\mathbf{x}}(t)$ and integrating in time we get the *energy relation*

$$E(t) + \int_0^t [\sum_{i=1}^N \frac{\mu_\beta}{m} |\dot{x}_i(s)| + \frac{\mu_g}{m} g(\dot{x}_i(t))\dot{x}_i(t)] ds = E(0), \tag{5}$$

where

$$E(t) = \frac{1}{2} \parallel \dot{\mathbf{x}}(t) \parallel^2 + \frac{k}{2m}\mathbf{Ax}(t) \cdot \mathbf{x}(t). \tag{6}$$

By (5), the trajectory $(\mathbf{x}(t), \dot{\mathbf{x}}(t))_{t \geq 0}$ is compact in $\mathbf{H}$, so, we can find $\alpha > 0$ such that $\mu_\beta |\dot{x}_i(t)| + \mu_g g(\dot{x}_i(t))\dot{x}_i(t) \geq \alpha |\dot{x}_i(t)|$ for $i = 1, \ldots, N$ and all $t \geq 0$. By (5), we conclude that $\dot{\mathbf{x}} \in L^1(\mathbb{R}_+)$ which leads to the existence of the limit $\mathbf{x}_\infty := \lim_{t\to+\infty} \mathbf{x}(t)$ and to $\lim_{t\to+\infty} \dot{\mathbf{x}}(t) = 0.\blacksquare$

In order to prove Theorem 2 it is useful to reformulate the problem in its nondimensional form

Lemma 1. *The change of scales* $\mathbf{x}(t) := \widetilde{\mathbf{x}}(\widetilde{t})x^*$, $\widetilde{t} = \frac{t}{t^*}$, $x^* = \frac{\mu_\beta}{k}$, $t^* = \sqrt{\frac{m}{k}}$, *transforms* $(\mathbf{P}_N)$ *in the nondimensional problem*

$$(\widetilde{\mathbf{P}}_N) \begin{cases} \ddot{\widetilde{\mathbf{x}}}(\widetilde{t}) + \mathbf{A}\widetilde{\mathbf{x}}(\widetilde{t}) + \mathbf{B}(\dot{\widetilde{\mathbf{x}}}(\widetilde{t})) + \frac{\mu_g}{\mu_\beta}\mathbf{G}(\frac{\mu_\beta}{\sqrt{mk}}\dot{\widetilde{\mathbf{x}}}(\widetilde{t})) \ni \mathbf{0}, \\ \widetilde{\mathbf{x}}(0) = \widetilde{\mathbf{x}}_0,\ \dot{\widetilde{\mathbf{x}}}(0) = \widetilde{\mathbf{v}}_0, \end{cases}$$

with $\widetilde{\mathbf{x}}_0 = \frac{k\mathbf{x}_0}{\mu_\beta}$ *and* $\widetilde{\mathbf{v}}_0 = \frac{\sqrt{mk}}{\mu_\beta}\mathbf{v}_0$.

Proof of Lemma 1. It is enough to check that $\dot{\mathbf{x}}(t) = \frac{x^*}{t^*}\frac{d\widetilde{\mathbf{x}}}{d\widetilde{t}}$ and to use that $\mathbf{B}(\theta\dot{\mathbf{x}}(t)) = \mathbf{B}(\dot{\mathbf{x}}(t))$ for any $\theta > 0.\blacksquare$

We come back to the proof of part i) of Theorem 2. In the following we shall identify $(\widetilde{\mathbf{P}}_N)$ with $(\mathbf{P}_N)$ if no confusion may arises. In view of Theorem 1 and Lemma 1, we have to prove that there exists $T_e \geq 0$ such that $\mathbf{x}(t) \equiv \mathbf{x}_\infty$ for all $t \geq T_e$. In what follows we shall adopt some notation similar to the introduced by Bamberger and Cabannes in [3]

$$\Delta_i(t) := (\mathbf{A}\mathbf{x}(t))_i \text{ and } \Delta_i^* := (\mathbf{A}\mathbf{x}_\infty)_i, \quad \text{for } i \in \{1, \dots, N\}.$$

We recall that, since $\mathbf{x}_\infty$ is an stationary point, we have $(\Delta_i^*)_{i=1}^N \in [-1,1]^N$. We need an auxiliary lemma describing the behavior of $\mathbf{x}(t)$ for large time. In the statement, the constants may depend on the initial data.

Lemma 2.

i) Suppose that for some $i \in 1, \dots, N$, $|\Delta_i^*| < 1$. *Then there exists* $T_i \geq 0$ *such that* $\forall t \geq T_i, \quad \dot{x}_i(t) = 0$.

ii) If, for some $i \in 1, \dots, N$, $\Delta_i^* = 1$ *(resp.* $\Delta_i^* = -1$*). Then there exists* $T_i \geq 0$ *such that* $\forall t \geq T_i, \quad \dot{x}_i(t) \leq 0 \quad$ *(resp.* $\dot{x}_i(t) \geq 0$*).*

Proof of Lemma 2. Let $0 < \delta << 1$ be fixed. By Theorem 1 we can find $t_0 \geq 0$ such that

$$\forall t \geq t_0, \quad |\Delta_i(t)| \leq (1-2\delta) \quad \text{and} \quad |g(\frac{\mu_\beta}{\sqrt{mk}}\,\dot{x}_i(t))| \leq \frac{\mu_\beta}{\mu_g}\delta. \tag{7}$$

If $\dot{x}_i(t_0) = 0$, we conclude that $x_i(t) \equiv x_i(t_0) = x_{\infty i}$ for all $t \geq t_0$ since $\Delta_i(t) \in [-1,1]$ for all $t \geq t_0$. If not, let $T = \sup\{s \geq t_0,\ |\dot{x}_i(t)| > 0\ \forall t \in [t_0, s[\}$. Multiplying the i-component of $(\mathbf{P}_N)$ by $\dot{x}_i(t)$ and using (7) we obtain

$$\frac{1}{2}\frac{d}{dt}(|\dot{x}_i(t)|^2) + \delta|\dot{x}_i(t)| \leq 0, \text{ for a.e. } t \in [t_0, T[. \tag{8}$$

Dividing (8) by $|\dot{x}_i(t)|$ we get

$$\frac{d}{dt}(|\dot{x}_i(t)|) + \delta \leq 0 \text{ for a.e. } t \in [t_0, T[. \tag{9}$$

Integrating, we see that $\dot{x}_i(t_0 + \frac{|\dot{x}_i(t_0)|}{\delta}) = 0$. Thus $T < +\infty$ and we conclude, as before, that $x_i(t) \equiv x_i(T) = x_{\infty i}$ for any $t \geq T$. To prove part ii) we consider, again,

$0 < \delta << 1$ and suppose that $\Delta_i^* = 1$ (the case $\Delta_i^* = -1$ is similar). By Theorem 1 we can find $t_0 \geq 0$ such that

$$\Delta_i(t) \geq \delta \quad \text{and} \quad |g(\frac{\mu_\beta}{\sqrt{mk}}\,\dot{x}_i(t))| \leq \frac{\mu_\beta}{\mu_g}\delta, \text{for a.e. } t \geq t_0. \tag{10}$$

Suppose that $\dot{x}_i(t_0) > 0$ and let $\tau = \sup\{s > t_0,\ \dot{x}_i(t) > 0 \quad \forall t \in [t_0, s[\}$. In $[t_0, \tau[$ we have

$$\ddot{x}_i(t) + \Delta_i(t) + 1 + \frac{\mu_g}{\mu_\beta} g(\frac{\mu_\beta}{\sqrt{mk}}\,\dot{x}_i(t)) = 0.$$

From (10), we get that $\ddot{x}_i(t) \leq -1$ in $[t_0, \tau[$ and by integration $\dot{x}_i(t) \leq \dot{x}_i(t_0) - (t - t_0)$ in $[t_0, \tau[$. Thus $\tau < +\infty$ and we conclude that we can find $T \geq t_0$ such that $\dot{x}_i(T) \leq 0$. Now suppose that there exists $t_1 > T$ such that $\dot{x}_i(t_1) > 0$. From the continuity of $\dot{x}_i$, there exists some interval $]t_2, t_3[$ with $t_2 > T$ and $\dot{x}_i(t_2) > 0$, where $\dot{x}_i$ is strictly increasing. In $]t_2, t_3[$ we have $\ddot{x}_i = -1 - \Delta_i - \frac{\mu_g}{\mu_\beta} g(\frac{\mu_\beta}{\sqrt{mk}}\,\dot{x}_i)$. Thus form the choice of δ, $\dot{x}$ is strictly decreasing in $]t_2, t_3[$, which is a contradiction. ■

Proof of Theorem 2 (continuation): i) Let $I^+ = \{i \in \{1, \ldots, N\},\ \Delta_i^* = 1\}$ and $I^- = \{i \in \{1, \ldots, N\},\ \Delta_i^* = -1\}$. In view of Lemma 2, we can find $T \geq 0$ such that for all $t \geq T$ we have that: i) $\forall i \in \{1, \ldots, N\},\quad g(\frac{\mu_\beta}{\sqrt{mk}}\,\dot{x}_i(t))\dot{x}_i(t) \leq 0$, ii) $\forall i \in I^+,\quad \dot{x}_i(t) \leq 0$, iii) $\forall i \in I^-,\quad \dot{x}_i(t) \geq 0$, and iv) $\forall i \notin I^+ \cup I^-,\quad \dot{x}_i(t) = 0$. We write the equations of $(\mathbf{P}_N)$ as

$$\ddot{x}_i(t) + \Delta_i(t) - \Delta_i^* + 1 + \beta(\dot{x}_i(t)) + \frac{\mu_g}{\mu_\beta}\, g(\frac{\mu_\beta}{\sqrt{mk}}\,\dot{x}_i(t)) \ni 0, \text{ for } i \in I^+, \tag{11}$$

(and analogy for $i \in I^-$). Multiplying by $\dot{x}_i(t)$ and summing over i, we get

$$\ddot{\mathbf{x}}(t) \cdot \dot{\mathbf{x}}(t) + \mathbf{A}(\mathbf{x}(t) - \mathbf{x}_\infty) \cdot \dot{\mathbf{x}}(t) + \frac{\mu_g}{\mu_\beta}\,\mathbf{G}(\frac{\mu_\beta}{\sqrt{mk}}\,\dot{\mathbf{x}}(t)) \cdot \dot{\mathbf{x}}(t) = 0, \forall t \geq T,$$

Integrating in time, we have

$$\parallel \dot{\mathbf{x}}(t) \parallel^2 + \mathbf{A}(\mathbf{x}(t) - \mathbf{x}_\infty) \cdot (\mathbf{x}(t) - \mathbf{x}_\infty) \geq \parallel \dot{\mathbf{x}}(T) \parallel^2 + \mathbf{A}(\mathbf{x}(T) - \mathbf{x}_\infty) \cdot (\mathbf{x}(T) - \mathbf{x}_\infty) \geq 0$$

Letting $t \to +\infty$ we obtain $\parallel \dot{\mathbf{x}}(T) \parallel^2 + \mathbf{A}(\mathbf{x}(T) - \mathbf{x}_\infty) \cdot (\mathbf{x}(T) - \mathbf{x}_\infty) = \mathbf{0}$. Since $\mathbf{A}$ is a positive definite matrix, we conclude that $\mathbf{x}(T) = \mathbf{x}_\infty$ and thus $\mathbf{x}(t) = \mathbf{x}_\infty$ for any $t \geq T$.

ii) Assume now that $g(r) = \lambda r$ with $\lambda \geq \frac{2\sqrt{\lambda_1 mk}}{\mu_g}$. In order to construct a solution of $(\mathbf{P}_N)$ which does not stabilize in finite time we search a particular solution of the vectorial linear ODE

$$\ddot{\mathbf{X}} + \mathbf{A}\mathbf{X} + \frac{\lambda \mu_g}{\sqrt{mk}}\,\dot{\mathbf{X}} = \mathbf{0}. \tag{12}$$

Since $\mathbf{A}$ is a symmetric definite positive matrix, we can find a matrix $\mathbf{P} \in \mathbb{R}^{N \times N}$ such that $\mathbf{A} = \mathbf{P}^T \mathrm{diag}(\lambda_1, \cdots, \lambda_N)\mathbf{P}$ with $0 < \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_N$ and $\mathbf{P}^T\mathbf{P} = \mathbf{I}$, the identity matrix. Writing $\mathbf{X} = \mathbf{P}^T\mathbf{Y}$, system (12) is equivalent to the system

$$\ddot{y}_i + \lambda_i y_i + \frac{\lambda \mu_g}{\sqrt{mk}}\,\dot{y}_i = 0 \quad \text{for } i = 1, \ldots, N. \tag{13}$$

The equation $\ddot{y}_1 + \lambda_1 y_1 + \frac{\lambda\mu_g}{\sqrt{mk}}\,\dot{y}_1 = 0$ admits a solution $y_1(t)$ such that $\dot{y}_1(t) < 0$ for all $t \geq 0$ since $\lambda \geq \frac{2\sqrt{\lambda_1 mk}}{\mu_g}$. We define $\mathbf{Y}(t) = (y_1(t), 0, \ldots, 0)$ which satisfies (13). Then, $\mathbf{X}(t) := \mathbf{P}^T\mathbf{Y}(t)$ satisfies (12) and is such that $\dot{x}_i(t)$ has a constant sign and never vanishes or $\dot{x}_i(t) \equiv 0$. If we denote by $\mathbf{\Delta}^*$ the constant vector of $\mathbb{R}^N$ defined by $\Delta_i^* = \beta_0(\dot{x}_i), \quad i = 1, \ldots, N$, with $\beta_0(r) = \beta(r)$ if $r \neq 0$ and $\beta_0(0) = 0$, and consider $\mathbf{x}_\infty$ as the solution of $\mathbf{A}\mathbf{x}_\infty = -\mathbf{\Delta}^*$. Summing $\mathbf{X}$ and $\mathbf{x}_\infty$, we get a solution of $(\mathbf{P}_N)$ which never stops.

iii) We suppose $N = 1$ (and take $A = 1$). The problem becomes

$$\ddot{x} + x + \beta(\dot{x}) + \frac{\mu_g}{\mu_\beta}\, g(\frac{\mu_\beta}{\sqrt{mk}}\,\dot{x}) \ni 0. \tag{14}$$

Firstly, suppose that $g'(0) < \frac{2\sqrt{mk}}{\mu_g}$. We want to prove that all solutions of (14) stabilize in finite time. In view of the previous steps, we only have to consider the case $|x(t)| \to 1$. By analogy, it is enough to consider the case $x(t) \to 1$. We know that there exists a time T such that $\dot{x}(t) \leq 0$ for all $t \geq T$. If the process does not stop at a time T, then there exists a $t_0 \geq T$ such that $\dot{x}(t_0) < 0$. Let $\tau = \sup\{t \geq t_0,\ \dot{x}(t) < 0\}$. Since g is regular near 0 and $g'(0) < \frac{2\sqrt{mk}}{\mu_g}$ we know by Hartman's Theorem ([16]) that the point $(1, 0)$ is a *center* or a *focus* for the equation

$$\ddot{u} + u - 1 + \frac{\mu_g}{\mu_\beta}\, g(\frac{\mu_\beta}{\sqrt{mk}}\,\dot{u}) = 0. \tag{15}$$

Since $x(t)$ satisfies this equation in (t_0, τ), we deduce that $\tau < \infty$ and $x(\tau) < 1$ with $\dot{x}(\tau) = 0$, thus the process stops at time τ which contradicts that $x(t) \to 1$ as $t \to +\infty$. If we assume, now, that $g'(0) \geq \frac{2\sqrt{mk}}{\mu_g}$, since g is regular near 0, by Hartman's Theorem, the point $(1, 0)$ is a *node* for equation (15) and we can find a solution $u(t)$ such that $\dot{u}(t) < 0$ for all $t \geq 0$. Such solution is also a solution of (14) which does not stabilize in any finite time.■

Remark 4. Similar results also hold for other N-dimensional systems arising when the spatial discretization of the wave equation is taken by finite elements instead by finite differences.

The dichotomy for the damped wave equation

As an illustration of possible extensions of *ii)* of Theorem 2 to other dynamical systems, we consider the damped wave equation in a bounded regular open set $\Omega \subset \mathbb{R}^N$

$$u_{tt} - \Delta u + \beta(u_t) + \lambda u_t \ni 0 \quad \text{in } \Omega \times (0, +\infty), \tag{16}$$

with Dirichlet boundary conditions $u(., t) = 0$ on $\partial\Omega$ for $t \in (0, +\infty)$. Let us assume that $\lambda \geq 2\sqrt{\lambda_1}$, with λ_1 the first eigenvalue of the operator $u \to -\Delta u$ associated to homogeneous Dirichlet boundary conditions. Then we can find some solutions of (16) which does not stabilize in any finite time and also some solutions which stabilizes in a finite time. We construct the first type of solutions in the form

$$u(x, t) = a(t)v(x) + \xi(x),$$

where v is a solution of the eigenvalue problem

$$\begin{cases} -\Delta v = \lambda_1 v & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega, \end{cases}$$

such that $v > 0$ in Ω, the function ξ is defined as the solution of the equation

$$\begin{cases} \Delta \xi = 1 & \text{in } \Omega, \\ \xi = 0 & \text{on } \partial\Omega, \end{cases}$$

and $a(t)$ is a solution of the ODE

$$\ddot{a} + \lambda_1 a + \lambda \dot{a} = 0, \tag{17}$$

such that $\dot{a}(t) > 0$ for any $t > 0$ (which is possible since $\lambda \geq 2\sqrt{\lambda_1}$). Then, we get a solution which does not stabilize in any finite time. By the contrary, if we choose $b(t)$ as a solution of (17) such that $\dot{b}(t) > 0$ for all $t \in [0, 1)$, $\dot{b}(1) = 0$ and $b(1) = K$ with $K = \frac{1}{\lambda_1 \|v\|_{L^\infty(\Omega)}}$ and take $a(t) = b(t)$ if $t \leq 1$ and $a(t) = K$ for $t \geq 1$ we get a solution which attains the stationary state $u_\infty(x) = Kv(x) + \xi(x)$ after $t = 1$.■

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