AX-SCHANUEL AND STRONG MINIMALITY FOR THE j-FUNCTION

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Abstract. Let $(K;+,\cdot,D,0,1)$ be a differentially closed field with field of constants C.

In the first part of the paper we explore the connection between Ax-Schanuel type theorems (predimension inequalities) for a differential equation $E(x,y)$ and the geometry of the fibres $U_s := \{y : E(s,y) \wedge y \notin C\}$ where s is a non-constant element. We show that certain types of predimension inequalities imply strong minimality and geometric triviality of U_s. Moreover, the induced structure on the Cartesian powers of U_s is given by special subvarieties. In particular, since the j-function satisfies an Ax-Schanuel inequality of the required form (due to Pila and Tsimerman), our results give a new proof for a theorem of Freitag and Scanlon stating that the differential equation of j defines a strongly minimal set with trivial geometry (which is not $\aleph_0$-categorical though).

In the second part of the paper we study strongly minimal sets in the j-reducts of dfferentially closed fields. Let $E_j(x,y)$ be the (two-variable) differential equation of the j-function. We prove a Zilber style classification result for strongly minimal sets in the reduct $\mathcal{K}_{E_j} := (K;+,\cdot,E_j)$ assuming an Existential Closedness (EC) conjecture for E_j. More precisely, assuming EC we show that in $\mathcal{K}_{E_j}$ all strongly minimal sets are geometrically trivial or non-orthogonal to C. The EC conjecture states roughly that if for a system of equations in terms of E_j having a solution does not contradict Ax-Schanuel then it does have a solution.

1. Introduction

Throughout the paper we let $\mathcal{K} = (K;+,\cdot,D,0,1)$ be a differentially closed field with field of constants C.

Let $E(x,y)$ be (the set of solutions of) a differential equation $f(x,y) = 0$ with rational (or, more generally, constant) coefficients. A general question that we are interested in is whether E satisfies an Ax-Schanuel type inequality. A motivating example is the exponential differential equation $Dy = yDx$. We know that (the original) Ax-Schanuel ([Ax71]) gives a predimension inequality (in the sense of Hrushovski [Hru93]) which governs the geometry of our equation. In this case the corresponding reduct of a differentially closed field can be reconstructed by a Hrushovski-style amalgamation-with-predimension construction ([Kir09]). Zilber calls this kind of predimension inequalities *adequate* (see [Asl17b, Asl18a] for a precise definition). This means that the reduct satisfies an existential closedness property which asserts roughly that a system of exponential equations which is not overdetermined has a solution. Being overdetermined means that the existence of a solution would contradict Ax-Schanuel. Thus, having an adequate Ax-Schanuel inequality for E will give us a complete understanding of its model theory. For more details on this and related problems see [Asl17b, Asl18a, Kir09, Zil04, Zil05]. Ax-Schanuel type statements can also be applied to diophantine geometry. Indeed, they can be used to prove a weak version of the famous Zilber-Pink conjecture in the appropriate setting (see [Zil02, PT16, Kir09, Asl18b]).

Thus, we want to classify differential equations of two variables with respect to the property of satisfying an Ax-Schanuel type inequality. The present work should be seen as a part of that more general project. In the first part of the paper we explore the connection between Ax-Schanuel type theorems (predimension inequalities) for a differential equation $E(x,y)$ and the geometry of the fibres of E. More precisely, given a predimension inequality (not necessarily adequate) for solutions of E of a certain type (which is of the form "td $-d$" where d is a

2010 *Mathematics Subject Classification.* 12H05, 03C60, 11F03, 12H20.

Key words and phrases. Ax-Schanuel theorem, Existential Closedness, j-function, strong minimality, geometric triviality.

dimension of trivial type) we show that the fibres of E are strongly minimal and geometrically trivial (after removing constant points). Moreover, the induced structure on the Cartesian powers of those fibres is given by *special* subvarieties.

One of the main results of the first part is as follows (for the definition of weakly special varieties see Section 3.1).

Theorem 1.1. *Let $E(x,y)$ be defined by $R(x,y,\partial_x y,\ldots,\partial_x^m y)=0$ where $\partial_x=\frac{D}{Dx}$ for a non-constant x and $R(X,\bar{Y})$ is an algebraic polynomial over C, irreducible over $C(X)^{\mathrm{alg}}$. Assume E satisfies the following Ax-Schanuel condition for a collection $\mathcal{P}$ of algebraic polynomials $P(X,Y)\in C[X,Y]$:*

let $x_1,\ldots,x_n,y_1,\ldots,y_n$ be non-constant elements of K with $E(x_i,y_i)$. If $P(y_i,y_j)\neq 0$ for all $P\in\mathcal{P}$ and $i\neq j$ then

$$\operatorname{td}_C C(x_1,y_1,\partial_{x_1}y_1,\ldots,\partial_{x_1}^{m-1}y_1,\ldots,x_n,y_n,\partial_{x_n}y_n,\ldots,\partial_{x_n}^{m-1}y_n)\geq mn+1.$$

Then for every $s\in K\setminus C$ the set $U_s:=\{y: E(s,y)\wedge y\notin C\}$ is strongly minimal with trivial geometry. Furthermore, every definable subset of a Cartesian power of U_s is a Boolean combination of weakly $\mathcal{P}$-special subvarieties.

In particular, let $F_j(y,Dy,D^2y,D^3y)=0$ be the differential equation of the j-function (see Section 2). Consider its two-variable version $E_j(x,y)$ given by $F_j(y,\partial_x y,\partial_x^2 y,\partial_x^3 y)=0$ (where, as above, $\partial_x=\frac{1}{Dx}\cdot D$). It is known (due to Pila and Tsimerman [PT16]) that E_j satisfies an Ax-Schanuel inequality of the above form where $\mathcal{P}$ is the collection of all modular polynomials. Hence the above result implies that the set $F_j(y,Dy,D^2y,D^3y)=0$ is strongly minimal and geometrically trivial thus giving a new proof for a theorem of Freitag and Scanlon [FS18]. In fact, the Pila-Tsimerman inequality is the main motivation for this paper.

Thus we get a necessary condition for E to satisfy an Ax-Schanuel inequality of the given form. This is a step towards the solution of the problem described above. In particular it gives rise to a converse problem: given a one-variable differential equation which is strongly minimal and geometrically trivial, can we say anything about the Ax-Schanuel properties of its two-variable analogue (see Section 3.4 for more details)?

On the other hand, understanding the structure of strongly minimal sets in a given theory is a central problem in geometric model theory. In DCF_0 there is a nice classification of strongly minimal sets. Namely, they satisfy the Zilber trichotomy, that is, such a set must be either geometrically trivial or non-orthogonal to a Manin kernel[1] (this is the locally modular non-trivial case) or non-orthogonal to the field of constants which corresponds to the non-locally modular case ([HuS93]). Hrushovski [Hru95] also gave a full characterisation of strongly minimal sets of order 1 proving that such a set is either non-orthogonal to the constants or it is trivial and $\aleph_0$-categorical. However there is no general classification of trivial strongly minimal sets of higher order and therefore we do not fully understand the nature of those sets. From this point of view the set J defined by the differential equation of j is quite intriguing since it is the first example of a trivial strongly minimal set in DCF_0 which is not $\aleph_0$-categorical. Before Freitag and Scanlon established those properties of J in [FS18], it was mainly believed that trivial strongly minimal sets in DCF_0 must be $\aleph_0$-categorical. The reason for this speculation was Hrushovski's aforementioned theorem on order 1 strongly minimal sets (and the lack of counterexamples).

Thus, the classification of strongly minimal sets in DCF_0 can be seen as another source of motivation for the work in this paper, where we show that these two problems (Ax-Schanuel type theorems and geometry of strongly minimal sets) are in fact closely related.

In the second part of the paper we use Ax-Schanuel for the j-function to classify strongly minimal sets in a "j-reduct" of a differentially closed field (this problem was asked by Zilber in a private communication). More precisely, the problem is to classify strongly minimal sets

[1] More precisely, it is non-orthogonal to the Manin kernel $A^{\#}$ of a simple abelian variety A of C-trace zero.

in the reduct $\mathcal{K}_{E_j} := (K; +, \cdot, E_j)$ where E_j is the two-variable differential equation of the j-function described above. It turns out that if we assume E_j satisfies an Existential Closedness statement ([Asl18a]), which essentially states that if for a system of equations in $\mathcal{K}_{E_j}$ having a solution does not contradict Ax-Schanuel then it does have a solution, then we can prove a dichotomy result for strongly minimal sets in $\mathcal{K}_{E_j}$.

Theorem 1.2. *Assume the Existential Closedness conjecture for E_j. Then in $\mathcal{K}_{E_j}$ all strongly minimal sets are geometrically trivial or non-orthogonal to C (the latter being definable in $\mathcal{K}_{E_j}$).*

The Existential Closedness conjecture is related to the question of adequacy of the Ax-Schanuel theorem for the j-function (see [Asl18a], we also give some details in Section 4.2). Adequacy means that the Ax-Schanuel inequality governs the geometry of the reduct, hence it is not surprising that it leads to a classification of strongly minimal sets there.

We also study strongly minimal sets in a more basic reduct, namely $\mathcal{K}_C := (K; +, \cdot, C)$ where C is the field of constants (this is just a pair of algebraically closed fields). Actually, this is the first example that we deal with in the second part of this paper. For this reduct we do not have any Ax-Schanuel type statement and we do not need one since it is quite easy to understand definable sets in such a structure. In this case we have the following result.

Theorem 1.3. *All strongly minimal sets in $\mathcal{K}_C$ are non-orthogonal to C.*

Most of our observations on pairs of algebraically closed fields are well known and we merely present our approach as a prelude to the aforementioned classification of strongly minimal sets in j-reducts.

The paper is organised as follows. In Section 2 we give a brief account of the j-function. Section 3 is the "first part" of the paper where we study strong minimality and geometric triviality of definable sets in DCF_0. Section 4, the "second part", is devoted to the classification of strongly minimal sets in j-reducts of DCF_0. Appendix A contains some preliminaries on strongly minimal sets.

Notation and conventions.

- The length of a tuple $\bar{a}$ will be denoted by $|\bar{a}|$.
- For fields $L \subseteq K$ the transcendence degree of K over L is denoted by $\mathrm{td}(K/L)$ or $\mathrm{td}_L K$. The algebraic locus (Zariski closure) of a tuple $\bar{a} \in K$ over L will be denoted by $\mathrm{Loc}_L(\bar{a})$ or $\mathrm{Loc}(\bar{a}/L)$.
- The algebraic closure of a field L is denoted by L^{alg}.
- Algebraic varieties defined over an (algebraically closed) field K will be identified with the sets of their K-rational points.
- In a differential field $(K; +, \cdot, D)$ and a non-constant element x the *differentiation with respect to* x is a derivation ∂_x of K defined by $\partial_x : y \to \frac{Dy}{Dx}$.
- For differential fields $L \subseteq K$ and a subset $A \subseteq K$ the differential subfield of K generated by L and A will be denoted by $L\langle A \rangle$, and $K\{X\}$ is the ring of differential polynomials over K.

2. Background on the j-function

The j-function is a modular function of weight 0 for the modular group $\mathrm{SL}_2(\mathbb{Z})$, which is defined and analytic on the upper half-plane $\mathbb{H} := \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$.

Let $\mathrm{GL}_2^+(\mathbb{Q})$ be the subgroup of $\mathrm{GL}_2(\mathbb{Q})$ consisting of matrices with positive determinant (this group acts on the upper half-plane via the linear fractional transformations). For $g \in \mathrm{GL}_2^+(\mathbb{Q})$ we let $N(g)$ be the determinant of g scaled so that it has relatively prime integral entries. For each positive integer N there is an irreducible polynomial $\Phi_N(X, Y) \in \mathbb{Z}[X, Y]$ such that whenever $g \in \mathrm{GL}_2^+(\mathbb{Q})$ with $N = N(g)$, the function $\Phi_N(j(z), j(gz))$ is identically zero. Conversely, if $\Phi_N(j(x), j(y)) = 0$ for some $x, y \in \mathbb{H}$ then $y = gx$ for some $g \in \mathrm{GL}_2^+(\mathbb{Q})$ with $N = N(g)$. The polynomials Φ_N are called *modular polynomials* (see [Lan73]). It is well known that $\Phi_1(X,Y) = X - Y$ and all the other modular polynomials are symmetric. Two elements $w_1, w_2 \in \mathbb{C}$ are called *modularly independent* if they do not satisfy any modular relation $\Phi_N(w_1,w_2) = 0$. This definition makes sense for arbitrary fields (of characteristic zero) as the modular polynomials have integer coefficients.

The j-function satisfies an order 3 algebraic differential equation over $\mathbb{Q}$, and none of lower order (i.e. its differential rank over $\mathbb{C}$ is 3). Namely, $F_j(j,j',j'',j''') = 0$ where

$$F_j(Y_0,Y_1,Y_2,Y_3) = \frac{Y_3}{Y_1} - \frac{3}{2}\left(\frac{Y_2}{Y_1}\right)^2 + \frac{Y_0^2 - 1968Y_0 + 2654208}{2Y_0^2(Y_0-1728)^2} \cdot Y_1^2.$$

Thus

$$F_j(Y,Y',Y'',Y''') = SY + R(Y)(Y')^2,$$

where S denotes the *Schwarzian derivative* defined by $SY = \frac{Y'''}{Y'} - \frac{3}{2}\left(\frac{Y''}{Y'}\right)^2$ and $R(Y) = \frac{Y^2-1968Y+2654208}{2Y^2(Y-1728)^2}$. All functions $j(gz)$ with $g \in \mathrm{SL}_2(\mathbb{C})$ satisfy this equation and all solutions are of that form (if one wants a solution to be defined on $\mathbb{H}$ then one takes $g \in \mathrm{SL}_2(\mathbb{R})$).

Here $'$ denotes the derivative of a complex function. Below when we work in an abstract differential field we will always denote its derivation by D and for an element a in that field $a', a'', \ldots$ will be some other elements and not necessarily the derivatives of a.

In an abstract differential field $(K;+,\cdot,D,0,1)$ the differential equation of j is the equation $F_j(y,Dy,D^2y,D^3y) = 0$. Consider its two-variable version.

$$f_j(x,y) := F_j\left(y, \partial_x y, \partial_x^2 y, \partial_x^3 y\right) = 0.$$

Theorem 2.1 (Ax-Schanuel for j, [PT16])**.** *Let $z_i, j_i \in K \setminus C$, $i = 1,\ldots,n$, be such that $f_j(z_i,j_i) = 0$. If j_i's are pairwise modularly independent then*

$$\mathrm{td}_C\, C(z_i, \partial_{z_i} j_i, \partial_{z_i}^2 j_i : 1 \le i \le n) \ge 3n+1.$$

3. Ax-Schanuel and geometry of strongly minimal sets in DCF_0

3.1. **Setup and main results.** Recall that $\mathcal{K} = (K;+,\cdot,D,0,1)$ is a differentially closed field with field of constants C. We may assume (without loss of generality) $\mathcal{K}$ is sufficiently saturated if necessary. Fix an element t with $Dt = 1$. Let $E(x,y)$ be (the set of solutions of) a differential equation $f(x,y) = 0$ with constant coefficients.

We give several definitions and then state the main results of the first part of the paper.

Definition 3.1. Let $\mathcal{P}$ be a non-empty collection of algebraic polynomials $P(X,Y) \in C[X,Y]$. We say two elements $a,b \in K$ are *$\mathcal{P}$-independent* if $P(a,b) \ne 0$ and $P(b,a) \ne 0$ for all $P \in \mathcal{P}$. The *$\mathcal{P}$-orbit* of an element $a \in K$ is the set $\{b \in K : P(a,b) = 0 \text{ or } P(b,a) = 0 \text{ for some } P \in \mathcal{P}\}$ (in analogy with a Hecke orbit). Also, $\mathcal{P}$ is said to be *trivial* if it consists only of the polynomial $X - Y$.

Recall that $f(x,y) = 0$ is the differential equation defining E and denote $m := \mathrm{ord}_Y f(X,Y)$ (the order of f with respect to Y).

Definition 3.2. We say $E(x,y)$ has the $\mathcal{P}$-AS (Ax-Schanuel with respect to $\mathcal{P}$) property if the following condition is satisfied.

Let $x_1,\ldots,x_n,y_1,\ldots,y_n$ be non-constant elements of K with $f(x_i,y_i) = 0$. If the y_i's are pairwise $\mathcal{P}$-independent then

$$\mathrm{td}_C\, C\left(x_1, y_1, \partial_{x_1} y_1, \ldots, \partial_{x_1}^{m-1} y_1, \ldots, x_n, y_n, \partial_{x_n} y_n, \ldots, \partial_{x_n}^{m-1} y_n\right) \ge mn+1. \tag{3.1}$$

We say E has the $\mathcal{P}$-ALW (Ax-Lindemann-Weierstrass with respect to $\mathcal{P}$) property if the inequality (3.1) is satisfied under an additional assumption $\mathrm{td}_C\, C(\bar{x}) = 1$.

The $\mathcal{P}$-AS property can be reformulated as follows: for any non-constant solutions (x_i,y_i) of E the transcendence degree in (3.1) is strictly bigger than m times the number of different $\mathcal{P}$-orbits of y_i's. Note that (3.1) is motivated by the known examples of Ax-Schanuel inequalities ([Ax71, PT16, Asl17a]).

Remark 3.3. Having the $\mathcal{P}$-AS property for a given equation E may force $\mathcal{P}$ to be "closed" in some sense. Firstly, $X - Y$ (or a polynomial divisible by $X - Y$) must be in $\mathcal{P}$. Secondly, if $P_1, P_2 \in \mathcal{P}$ then $P_1(y_1, y_2) = 0$, $P_2(y_2, y_3) = 0$ impose a relation on y_1 and y_3 given by $Q(y_1, y_3) = 0$ for some polynomial Q. Then the $\mathcal{P}$-AS property may fail if $Q \notin \mathcal{P}$. In that case one has to add Q to $\mathcal{P}$ in order to allow the possibility of an Ax-Shcanuel property with respect to $\mathcal{P}$.

Similar conditions on $\mathcal{P}$ are required in order for $\mathcal{P}$-independence to define a dimension function (number of distinct $\mathcal{P}$-orbits) of a pregeometry (of trivial type), which would imply that the $\mathcal{P}$-AS property is a predimension inequality. Note that the collection of modular polynomials has all those properties. However, the shape of $\mathcal{P}$ is not important for our results since we assume that a given equation E has the $\mathcal{P}$-AS property.

Definition 3.4. A $\mathcal{P}$-*special variety* (in K^n for some n) is an irreducible (over C) component of a Zariski closed set in K^n defined by a finite collection of equations of the form $P_{ik}(y_i, y_k) = 0$ for some $P_{ik} \in \mathcal{P}$. For a subfield $L \subseteq K$ a *weakly* $\mathcal{P}$-*special variety over* L is an irreducible (over L^{alg}) component of a Zariski closed set in K^n defined by a finite collection of equations of the form $P_{ik}(y_i, y_k) = 0$ and $y_i = a$ for some $P_{ik} \in \mathcal{P}$ and $a \in L^{\text{alg}}$. For a definable set V, a (weakly) $\mathcal{P}$-*special subvariety* (over L) of V is an intersection of V with a (weakly) $\mathcal{P}$-special variety (over L).

A $\mathcal{P}$-special variety S may have a constant coordinate defined by an equation $P(y_i, y_i) = 0$ for some $P \in \mathcal{P}$. If no coordinate is constant on S then it is said to be strongly $\mathcal{P}$-special.

Let $C_0 \subseteq C$ be the subfield of C generated by the coefficients of f and let $K_0 = C_0\langle t\rangle = C_0(t)$ be the (differential) subfield of K generated by C_0 and t (clearly U is defined over K_0). We fix K_0 and work over it (in other words we expand our language with new constant symbols for elements (generators) of K_0).

Now we can formulate one of our main results (see Appendix A for definitions of geometric triviality and strict disintegratedness).

Theorem 3.5. *Assume $E(x, y)$ satisfies the $\mathcal{P}$-AS property for some $\mathcal{P}$. Assume further that the differential polynomial $g(Y) := f(t, Y)$ is absolutely irreducible.*[2] *Then*

- *$U := \{y : g(y) = 0 \wedge Dy \neq 0\}$ is strongly minimal with trivial geometry.*
- *If, in addition, $\mathcal{P}$ is trivial then U is strictly disintegrated and hence it has $\aleph_0$-categorical induced structure.*
- *All definable subsets of U^n over a differential field $L \supseteq K_0$ are Boolean combinations of weakly $\mathcal{P}$-special subvarieties over L.*

Remark 3.6. If the polynomials from $\mathcal{P}$ have rational coefficients then $\mathcal{P}$-special varieties are defined over $\mathbb{Q}^{\text{alg}}$. Furthermore, if E satisfies the $\mathcal{P}$-AS property then $U \cap C(t)^{\text{alg}} = \emptyset$ and so $\mathcal{P}$-special subvarieties of U over $C(t)$ are merely $\mathcal{P}$-special subvarieties (over C).

As the reader may guess and as we will see in the proof, this theorem holds under weaker assumptions on E. Namely, it is enough to require that (3.1) hold for $x_1 = \ldots = x_n = t$ which is a weak form of the $\mathcal{P}$-ALW property. However, we prefer the given formulation of Theorem 3.5 since the main object of our interest is the Ax-Schanuel inequality (for E).

Further, we deduce from Theorem 3.5 that if E has some special form, then all fibres $E(s, y)$ for a non-constant $s \in K$ have the above properties (over $C_0\langle s\rangle$).

Theorem 3.7. *Let $E(x, y)$ be defined by $R(x, y, \partial_x y, \ldots, \partial_x^m y) = 0$ where $R(X, \bar{Y})$ is an algebraic polynomial over C, irreducible over $C(X)^{\text{alg}}$ (as a polynomial of $\bar{Y}$). Assume $E(x, y)$ satisfies the $\mathcal{P}$-AS property for some $\mathcal{P}$ and let $s \in K$ be a non-constant element. Then*

- *$U_s := \{y : E(s, y) \wedge Dy \neq 0\}$ is strongly minimal with trivial geometry.*
- *If, in addition, $\mathcal{P}$ is trivial then any distinct non-algebraic (over $C_0\langle s\rangle$) elements are independent and U_s is $\aleph_0$-categorical.*

[2]Recall that t is an element with $Dt = 1$.

- *All definable subsets of U_s^n over a differential field $L \supseteq C_0\langle s\rangle$ are Boolean combinations of weakly $\mathcal{P}$-special subvarieties over L.*

Remark 3.8. Since $U_s \cap C = \emptyset$, in Theorems 3.5 and 3.7 the induced structure on U_s^n is actually given by *strongly special* subvarieties (over L), which means that we do not allow any equation of the form $y_i = c$ for c a constant. In particular we also need to exclude equations of the form $P(y_i, y_i) = 0$ for $P \in \mathcal{P}$.

We also prove a generalisation of Theorem 3.5.

Theorem 3.9. *Assume $E(x, y)$ satisfies the $\mathcal{P}$-AS property and let $p(Y) \in C(t)[Y] \setminus C$, $q(Y) \in C[Y] \setminus C$ be such that the differential polynomial $f(p(Y), q(Y))$ is absolutely irreducible. Then the set*

$$U_{p,q} := \{y : E(p(y), q(y)) \wedge y \notin C\}$$

is strongly minimal and geometrically trivial.

As an application of Theorem 3.5 we obtain a result on the differential equation of the j-function which was established by Freitag and Scanlon in [FS18].

Theorem 3.10 ([FS18])**.** *The set $J \subseteq K$ defined by $F_j(y, Dy, D^2y, D^3y) = 0$ is strongly minimal with trivial geometry. Furthermore, J is not $\aleph_0$-categorical.*

Strong minimality and geometric triviality of J follow directly from Theorem 3.5 combined with the Ax-Schanuel theorem for j (see Section 2). Moreover, as we pointed out above in the proof of Theorem 3.5 we only use the $\mathcal{P}$-ALW property. In the case of j it is equivalent to Pila's modular Ax-Lindemann-Weierstrass with derivatives theorem ([Pil13]). Of course the "furthermore" clause does not follow from Theorem 3.5 but it is not difficult to prove. Theorem 3.5 also gives a characterisation of the induced structure on the Cartesian powers of J. Again, that result can be found in [FS18].

The proof of Theorem 3.10 by Freitag and Scanlon is based on Pila's modular Ax-Lindemann-Weierstrass with derivatives theorem along with Seidenberg's embedding theorem and Nishioka's result on differential equations satisfied by automorphic functions ([Nis89]). They also make use of some tools of stability theory such as the "Shelah reflection principle". However, as one may guess, we cannot use Nishioka's theorem (or some analogue of that) in the proof of 3.5 since we do not know anything about the analytic properties of the solutions of our differential equation. Thus, we show in particular that Theorem 3.10 can be deduced from Pila's result abstractly. The key point that makes this possible is stable embedding, which means that if $\mathcal{M}$ is a model of a stable theory and $X \subseteq M$ is a definable set over some $A \subseteq M$ then every definable (with parameters from M) subset of X^n can in fact be defined with parameters from $X \cup A$ (see Appendix A).

Let us stress once more that the set J is notable for being the first example of a strongly minimal set (definable in DCF_0) with trivial geometry that is not $\aleph_0$-categorical. Indeed the aforementioned result of Hrushovski on strongly minimal sets of order 1 led people to believe that all geometrically trivial strongly minimal sets must be $\aleph_0$-categorical. Nevertheless, it is not true as the set J illustrates.

3.2. **Proofs of the main results.**

Proof of Theorem 3.5. Taking $x_1 = \ldots = x_n = t$ in the $\mathcal{P}$-AS property we get the following weak version of the $\mathcal{P}$-ALW property for U which in fact is enough to prove Theorem 3.5.

Lemma 3.11. *$\mathcal{P}$-AS implies that for any pairwise $\mathcal{P}$-independent elements $u_1, \ldots, u_n \in U$ the elements $\bar{u}, D\bar{u}, \ldots, D^{m-1}\bar{u}$ are algebraically independent over $C(t)$ and hence over K_0.*

We show that every definable (possibly with parameters) subset V of U is either finite or co-finite. Since U is defined over K_0, by stable embedding there is a finite subset $A = \{a_1, \ldots, a_n\} \subseteq U$ such that V is defined over $K_0 \cup A$. It suffices to show that U realises a unique non-algebraic type over $K_0 \cup A$, i.e. for any $u_1, u_2 \in U \setminus \operatorname{acl}(K_0 \cup A)$ we have $\operatorname{tp}(u_1/K_0 \cup A) = \operatorname{tp}(u_2/K_0 \cup A)$. Let $u \in U \setminus \operatorname{acl}(K_0 \cup A)$. We know that $\operatorname{acl}(K_0 \cup A) = K_0\langle A\rangle^{\mathrm{alg}} = K_0(\bar{a}, D\bar{a}, \ldots, D^{m-1}\bar{a})^{\mathrm{alg}}$. Since $u \notin K_0\langle A\rangle^{\mathrm{alg}}$, u is transcendental over $K_0(A)$ and hence it is $\mathcal{P}$-independent from each a_i. We may assume without loss of generality that a_i's are pairwise $\mathcal{P}$-independent (otherwise we could replace A by a maximal pairwise $\mathcal{P}$-independent subset). Applying Lemma 3.11 to $\bar{a}, u$, we deduce that $u, Du, \ldots, D^{m-1}u$ are algebraically independent over $K_0\langle A\rangle$. Hence $\operatorname{tp}(u/K_0 \cup A)$ is determined uniquely (axiomatised) by the set of formulae

$$\{g(y) = 0\} \cup \{h(y) \neq 0 : h(Y) \in K_0\langle A\rangle\{Y\},\ \operatorname{ord}(h) < m\}$$

(Recall that g is absolutely irreducible and hence it is irreducible over any field). In other words $g(Y)$ is the minimal differential polynomial of u over $K_0\langle A\rangle$.

Thus U is strongly minimal. A similar argument shows also that if $A \subseteq U$ is a (finite) subset and $u \in U \cap \operatorname{acl}(K_0 A)$ then there is $a \in A$ such that $u \in \operatorname{acl}(K_0 a)$. This proves that U is geometrically trivial.

If $\mathcal{P}$ is trivial then distinct elements of U are independent, hence U is strictly disintegrated.

The last part of Theorem 3.5 follows from the following lemma.

Lemma 3.12. *Every irreducible (relatively) Kolchin closed (over $C(t)$) subset of U^n is a $\mathcal{P}$-special subvariety of U^n.*

Proof. Let $V \subseteq U^n$ be an irreducible relatively closed subset (i.e. it is the intersection of U^n with an irreducible Kolchin closed set in K^n). Pick a generic point $\bar{v} = (v_1, \ldots, v_n) \in V$ and let $W \subseteq K^n$ be the Zariski closure of $\bar{v}$ over C. Let $d := \dim W$ and assume $v_1, \ldots, v_d$ are algebraically independent over C. Then $v_i \in (C(v_1, \ldots, v_d))^{\mathrm{alg}}$ for each $i = d+1, \ldots, n$. By Lemma 3.11 each v_i with $i > d$ must be in a $\mathcal{P}$-relation with some v_{k_i} with $k_i \leq d$. Let $P_i(v_i, v_{k_i}) = 0$ for $i > d$. The algebraic variety defined by the equations $P_i(y_i, y_{k_i}) = 0,\ i = d+1, \ldots, n$, has dimension d and contains W. Therefore W is a component of that variety and so it is a $\mathcal{P}$-special variety.

We claim that $W \cap U^n = V$. Since $v_1, \ldots, v_d \in U$ are algebraically independent over C, by Lemma 3.11 $\bar{v}, D\bar{v}, \ldots, D^{m-1}\bar{v}$ are algebraically independent over $C(t)$. Moreover, the (differential) type of each $v_i,\ i > d$, over $v_1, \ldots, v_d$ is determined uniquely by an irreducible algebraic equation. Therefore $\operatorname{tp}(\bar{v}/C(t))$ is axiomatised by formulas stating that $\bar{v}$ is Zariski generic in W and belongs to U^n. In other words $\bar{v}$ is Kolchin generic in $W \cap U^n$. Now V and $W \cap U^n$ are both equal to the Kolchin closure of $\bar{v}$ inside U^n and hence they are equal. $\square$

Thus definable (over $C(t)$) subsets of U^n are Boolean combinations of special subvarieties. Now let $L \subseteq K$ be an arbitrary differential subfield over which U is defined. Then definable subsets of U^n over L can be defined with parameters from $\tilde{L} = K_0 \cup (U \cap L^{\mathrm{alg}})$ (see Appendix A). Then Lemma 3.12 implies that irreducible Kolchin closed subsets of U^n defined over $\tilde{L}$ are weakly $\mathcal{P}$-special subvarities of U^n over L.

Finally, note that since U does not contain any algebraic elements over $C(t)$, the type of any element $u \in U$ over $C(t)$ is isolated by the formula $f(t, y) = 0 \wedge Dy \neq 0$.

Proof of Theorem 3.9. We argue as above and show that for a finite set $A = \{a_1, \ldots, a_n\} \subseteq U_{p,q}$ there is a unique non-algebraic type over $K_0\langle A\rangle$ realised in $U_{p,q}$. Here we will use the full Ax-Lindemann-Weierstrass.

If $u \in U_{p,q} \setminus (K_0\langle A\rangle)^{\mathrm{alg}}$ then $q(u)$ is transcendental over $K_0(A)$ and so $q(u)$ is $\mathcal{P}$-independent from each $q(a_i)$. Moreover, we may assume $\{q(a_1), \ldots, q(a_n)\}$ is $\mathcal{P}$-independent. Then by the $\mathcal{P}$-AS property

$$\operatorname{td}_C C\left(p(u), q(u), \ldots, \partial_{p(u)}^{m-1}q(u), p(a_i), q(a_i), \ldots, \partial_{p(a_i)}^{m-1}q(a_i)\right)_{i=1,\ldots,n} \geq m(n+1)+1.$$

But then

$$\operatorname{td}_C C\left(t, u, Du, \ldots, D^{m-1}u, a_i, Da_i, \ldots, D^{m-1}a_i\right)_{i=1,\ldots,n} \geq m(n+1)+1,$$

and hence $u, Du, \ldots, Du$ are algebraically independent over $K_0\langle A\rangle$. This determines the type $\mathrm{tp}(u/K_0A)$ uniquely as required. It also shows triviality of the geometry.

Proof of Theorem 3.7. Consider the differentially closed field $\mathcal{K}_s = (K; +, \cdot, \partial_s, 0, 1)$. The given form of the differential equation E implies that U_s is defined over $C_0(s)$ in $\mathcal{K}_s$. However, in general it may not be defined over $C_0(s)$ in $\mathcal{K}$, it is defined over $C_0\langle s\rangle = C_0(s, Ds, D^2s, \ldots)$. As $s \notin C$, it is transcendental over C and so $R(s, \bar{Y})$ is irreducible over $C(s)^{\mathrm{alg}}$. Therefore $R(s, Y, \partial_s Y, \ldots, \partial_s^m Y)$ is absolutely irreducible. Since $\partial_s s = 1$, we know by Theorem 3.5 that U_s is strongly minimal in $\mathcal{K}_s$. On the other hand the derivations ∂_s and D are inter-definable (with parameters) and so a set is definable in $\mathcal{K}$ if and only if it is definable in $\mathcal{K}_s$ (possibly with different parameters). This implies that every definable subset of U_s in $\mathcal{K}$ is either finite or co-finite, hence it is strongly minimal.

Further, Theorem 3.5 implies that U_s is geometrically trivial over $C_0(s)$ in $\mathcal{K}_s$. By Theorem A.2, U_s is also geometrically trivial over $C_0\langle s\rangle$ in $\mathcal{K}_s$. On the other hand for any subset $A \subseteq U_s$ the algebraic closure of $C_0\langle s\rangle \cup A$ is the same in $\mathcal{K}$ and $\mathcal{K}_s$. This implies geometric triviality of U_s in $\mathcal{K}_s$.

The same argument (along with the remark after Theorem A.2) shows that the second and the third parts of Corollary 3.7 hold as well.

3.3. **Application to the j-function.** Recall that the differential equation of the j-function is given by

$$F_j(y, Dy, D^2y, D^3y) = Sy + R(y)(Dy)^2 = 0, \tag{3.2}$$

where S denotes the Schwarzian derivative defined by $Sy = \frac{D^3y}{Dy} - \frac{3}{2}\left(\frac{Dy}{Dy}\right)^2$ and $R(y) = \frac{y^2-1968y+2654208}{2y^2(y-1728)^2}$. Let J be the set defined by (3.2). Note that F_j is not a polynomial but a rational function. In particular constant elements do not satisfy (3.2) for Sy is not defined for a constant y. We can multiply our equation through by a common denominator and make it into a polynomial equation

$$F_j^*(y, Dy, D^2y, D^3y) = q(y)DyD^3y - \frac{3}{2}q(y)(D^2y)^2 + p(y)(Dy)^4 = 0, \tag{3.3}$$

where p and q are respectively the numerator and the denominator of R. Let J^* be the set defined by (3.3). It is not strongly minimal since C is a definable subset. However, as we will see shortly, $J = J^* \setminus C$ is strongly minimal and $\mathrm{MR}(J^*) = 1,\ \mathrm{MD}(J^*) = 2$. Thus whenever we speak of the formula $F_j(y, Dy, D^2y, D^3y) = 0$ (which, strictly speaking, is not a formula in the language of differential rings) we mean the formula $F_j^*(y, Dy, D^2y, D^3y) = 0 \wedge Dy \neq 0$.

Let $\Phi := \{\Phi_N(X, Y) : N > 0\}$ be the collection of modular polynomials. Then two elements are modularly independent iff they are Φ-independent. For an element $a \in K$ its *Hecke orbit* is its Φ-orbit.

Consider the two-variable analogue of the equation (3.3):

$$f_j^*(x, y) := F_j^*(y, \partial_x y, \partial_x^2 y, \partial_x^3 y) = 0. \tag{3.4}$$

Theorem 2.1 states that (3.4) has the Φ-AS property. As a consequence of Theorems 3.5 and 2.1 we get strong minimality and geometric triviality of J (note that $F_j^*(Y_0, Y_1, Y_2, Y_3)$ is absolutely irreducible for it depends linearly on Y_3) which was first established by Freitag and Scanlon in [FS18].

Lemma 3.11 for j is of course a special case of the Ax-Schanuel theorem for j. Nevertheless it can also be deduced from Pila's modular Ax-Lindemann-Weierstrass with derivatives theorem ([Pil13]) by employing Seidenberg's embedding theorem. Therefore only Pila's theorem is enough to prove strong minimality and geometric triviality of J. Moreover, Theorem 3.7 shows that all non-constant fibres of (3.4) are strongly minimal and geometrically trivial (after removing constant points) and the induced structure on the Cartesian powers of those fibres is given by (strongly) special subvarieties. Note that it is proven in [FS18] that the sets $F_j(y, Dy, D^2y, D^3y) = a$ have the same properties for any a.

Remark 3.13. To complete the proof of Theorem 3.10, that is, to show that J is not $\aleph_0$-categorical, one argues as follows (see [FS18]). The Hecke orbit of an element $j \in J$ is infinite and is contained in J. Therefore J realises infinitely many algebraic types over an arbitrary element $j \in J$ and hence is not $\aleph_0$-categorical.

3.4. **Some remarks.** An interesting question is whether there are differential equations with the $\mathcal{P}$-AS property with trivial $\mathcal{P}$. As we showed here, if $E(x, y)$ has such a property then the corresponding U (and other fibres too) must be strongly minimal and strictly disintegrated. There are quite a few examples of this kind of strongly minimal sets in DCF_0. The two-variable versions of those equations will be natural candidates of equations with the required $\mathcal{P}$-AS property. (Note that since in our proofs we only used the $\mathcal{P}$-ALW part of the $\mathcal{P}$-AS property, it would be more reasonable to expect those equations to satisfy the $\mathcal{P}$-ALW property for trivial $\mathcal{P}$. However, as it was mentioned earlier, we are mainly interested in the $\mathcal{P}$-AS properties of differential equations.)

For example, the geometry of the sets of the form $Dy = r(y)$, where r is a rational function over C, is well understood. The nature of the geometry is determined by the partial fraction decomposition of $1/r$. As an example consider the equation

$$Dy = \frac{y}{1+y}. \tag{3.5}$$

One can show that it defines a strictly disintegrated strongly minimal set ([Mar05]). The two variable analogue of this equation is

$$\partial_x y = \frac{y}{1+y}. \tag{3.6}$$

But this is equivalent to the equation $\frac{Dy}{y} = D(x-y)$. Denoting $z = x-y$ we get the exponential differential equation $Dy = yDz$. It is easy to deduce from this that (3.6) does not satisfy the $\mathcal{P}$-AS property with any $\mathcal{P}$ (it satisfies a version of the original exponential Ax-Schanuel inequality though). Indeed, the fibre of (3.6) above $x = t$ is of trivial type but the section by $x = t + y$ is non-orthogonal to C. So according to Theorem 3.9 the equation (3.6) does not satisfy any $\mathcal{P}$-AS property. Clearly, all the sets $Dy = r(y)$ can be treated in the same manner and hence they are not appropriate for our purpose. Thus, one needs to look at the behaviour of all the sets $E(p(y), q(y))$, and if they happen to be trivial strongly minimal sets then one can hope for a $\mathcal{P}$-AS inequality.

The classical Painlevé equations define strongly minimal and strictly disintegrated sets as well. For example, let us consider the first Painlevé equation $D^2y = 6y^2 + t$. Strong minimality and algebraic independence of solutions of this equation were shown by Nishioka in [Nis04] (note that strong minimality was discovered earlier by Kolchin (see [Mar05])). We consider its two-variable version

$$\partial_x^2 y = 6y^2 + x. \tag{3.7}$$

The goal is to find an Ax-Schanuel inequality for this equation. Observe that (3.7) does not satisfy the $\mathcal{P}$-AS property with trivial $\mathcal{P}$. Indeed, if ζ is a fifth root of unity then the transformation $x \mapsto \zeta^2 x,\ y \mapsto \zeta y$ sends a solution of (3.7) to another solution. If one believes these are the only relations between solutions of the above equation, then one can conjecture the following.

Conjecture 3.14 (Ax-Schanuel for the first Painlevé equation)**.** *If* $(x_i, y_i),\ i = 1, \ldots, n,$ *are solutions to the equation* (3.7) *and* $(x_i/x_j)^5 \neq 1$ *for* $i \neq j$ *then*

$$\mathrm{td}(\bar{x}, \bar{y}, \partial_{\bar{x}}\bar{y}) \geq 2n + 1.$$

One could in fact replace x's with y's in the condition $(x_i/x_j)^5 \neq 1$ as those are equivalent. Hence the above conjecture states that (3.7) has the $\{X^5 - Y^5\}$-AS property.

Nagloo and Pillay showed in [NP14] that the other generic Painlevé equations define strictly disintegrated strongly minimal sets as well. So we can analyse relations between solutions of their two-variable analogues and ask similar questions for them too.

4. Strongly minimal sets in j-reducts of DCF_0

First we study strongly minimal sets in pairs of algebraically closed fields. It will serve as a simple example of the methods that we are going to use in j-reducts.

4.1. **Pairs of algebraically closed fields.** Model theory of pairs of algebraically closed fields is well studied (see, for example, [AvdD16, Kei64]). Therefore, most of the results of this section are well known.

Let $\mathcal{K}_C := (K; +, \cdot, C)$ be an algebraically closed field of characteristic 0 with a distinguished algebraically closed subfield C (C is a unary predicate in the language). It is easy to prove that this structure is ω-stable of Morley rank ω. We assume $\mathcal{K}_C$ is sufficiently saturated.

Let $\bar{a} \in K^m$ and $b \in K$.

Lemma 4.1. $\mathrm{MR}(b/\bar{a}) < \omega$ *iff* $b \in C(\bar{a})^{\mathrm{alg}}$.

Proof. If b is transcendental over $C(\bar{a})$ then for any $b' \notin C(\bar{a})^{\mathrm{alg}}$ there is a field automorphism of K fixing $C(\bar{a})$ pointwise and mapping b to b'. In particular, it is an automorphism of $\mathcal{K}_C$ and so $\mathrm{tp}(b/\bar{a}) = \mathrm{tp}(b'/\bar{a})$, and this type is the generic type over $\bar{a}$. $\square$

Now let $b \in C(\bar{a})^{\mathrm{alg}}$. Then for some polynomial p the equality $p(\bar{a}, \bar{c}, b) = 0$ holds for some finite tuple $\bar{c} \in C^l$. Let $W := \mathrm{Loc}_{\mathbb{Q}(\bar{a})}(\bar{c}) \subseteq K^l$ be the algebraic locus (Zariski closure) of $\bar{c}$ over $\mathbb{Q}(\bar{a})$. For every proper subvariety $U \subsetneq W$ defined over $\bar{a}$ consider the formula

$$\varphi_U(y) = \exists \bar{x}(\bar{x} \in C^l \cap (W \setminus U) \wedge p(\bar{a}, \bar{x}, y) = 0). \tag{4.8}$$

Notice that for every $U \subsetneq W$ the formula $\varphi_U(b)$ holds. Observe also that the set $C^l \cap (W(K) \setminus U(K))$, being a subset of C^l, is actually definable with parameters from C. This follows from the stable embedding property.

Proposition 4.2. *The collection of all formulas* $\varphi_U(y)$ *determines (axiomatises)* $\mathrm{tp}(b/\bar{a})$.

Proof. Assume $b' \models \varphi_U(y)$ for all $U \subsetneq W$. The collection of formulas

$$\{\bar{x} \in C^l \cap (W \setminus U) \wedge p(\bar{a}, \bar{x}, b') = 0 : U \subsetneq W\}$$

(over $\bar{a}, b'$) is finitely satisfiable so it has a realisation $\bar{c}'$. Evidently $\bar{c}'$ is generic in W over $\bar{a}$. Therefore there is an automorphism π of $C(\bar{a})$ which fixes $\bar{a}$ pointwise, fixes C setwise and sends $\bar{c}$ to $\bar{c}'$. This automorphism can be extended to an automorphism of $\mathcal{K}_C$ which sends b to b'. $\square$

Remark 4.3. This shows, in particular, that the first-order theory[3] of $\mathcal{K}_C$ is nearly model complete, that is, every formula is equivalent to a Boolean combination of existential formulas modulo that theory. One can also show that it is not model complete. Indeed, pick three algebraically independent (over $\mathbb{Q}$) elements a, b, x and set $y := ax + b$. Let

$$C_0 := \mathbb{Q}^{\mathrm{alg}},\ C_1 := \mathbb{Q}(a,b)^{\mathrm{alg}},\ K_0 := \mathbb{Q}(x,y)^{\mathrm{alg}},\ K_1 := \mathbb{Q}(a,b,x)^{\mathrm{alg}}.$$

Then $K_0 \cap C_1 = C_0$ so $(K_0, C_0) \subseteq (K_1, C_1)$ but the extension is not elementary since the formula $\exists u, v \in C(y = ux + v)$ (with parameters x, y) holds in (K_1, C_1) but not in (K_0, C_0). Note that this argument is adapted from a standard proof of non-modularity of algebraically closed fields of transcendence degree at least 3.

Theorem 4.4. *A strongly minimal set definable in* $\mathcal{K}_C$ *is non-orthogonal to* C.

[3] This theory is axiomatised by axiom schemes stating that K is an algebraically closed field and C is an algebraically closed subfield.

Proof. Let $S \subseteq K$ be strongly minimal defined by some formula φ_U (a conjucntion of formulas of the form (4.8) is again of the same form). Then $S \subseteq C(\bar{a})^{\mathrm{alg}} \subseteq \mathrm{acl}(C \cup \bar{a})$ and therefore $S \not\perp C$. □

Remark 4.5. Let $S \subseteq K$ be strongly minimal defined by some formula φ_U. As we pointed out above $V := W(K) \setminus U(K) \cap C^l$ is defined over C. So V is a constructible set over C. Define an equivalence relation $E \subseteq V \times V$ by

$$\bar{c}_1 E \bar{c}_2 \text{ iff } \forall y (p(\bar{a}, \bar{c}_1, y) = 0 \leftrightarrow p(\bar{a}, \bar{c}_2, y) = 0).$$

By the stable embedding property E is definable in the pure field structure of C. Moreover, there is a natural finite-to-one map from S to V/E. By elimination of imaginaries in algebraically closed fields V/E can be regarded as a constructible set in some Cartesian power C^k. The latter must have dimension 1 since S is strongly minimal. Thus, in the formula φ_U we may assume that the constants live on a curve defined over C. This gives a characterisation of strongly minimal formulas.

4.2. **Predimension for the differential equation of the j-function.** Now we study the differential equation of the j-function. This and the next subsections are preliminary. The reader is referred to [Asl18a] for details and proofs of the results.

Let $f_j(x, y) = 0$ be the two-variable differential equation of the j-function (see Section 2). We consider a binary predicate $E_j(x, y)$ which will be interpreted in a differential field as the set of solutions of the equation $f_j(x, y) = 0$. This equation excludes the possibility of x or y being a constant. However, if we multiply $f_j(x, y)$ by a common denominator and make it a differential polynomial then x and y would be allowed to be constants as well. So we add C^2 to E_j, i.e. any pair of constants is in E_j. Further, let $E_j^{\times}$ be the set of all E_j-points with no constant coordinates.

The following is an immediate consequence of Theorem 2.1.

Theorem 4.6 (Ax-Schanuel without derivatives)**.** *If z_i, j_i are non-constant elements in a differential field K with $f_j(z_i, j_i) = 0$, then*

$$\mathrm{td}_C C(\bar{z}, \bar{j}) \geq n + 1,$$

unless for some $N, i \neq k$ we have $\Phi_N(j_i, j_k) = 0$.

Definition 4.7. The theory T_j^0 consists of the following first-order statements about a structure K in the language $\mathfrak{L}_j := \{+, \cdot, E_j, 0, 1\}$.

- A1 K is an algebraically closed field of characteristic 0.
- A2 $C := C_K = \{c \in K : E_j(0, c)\}$ is an algebraically closed subfield. Further, $C^2 \subseteq E_j(K)$ and if $(z, j) \in E_j(K)$ and one of z, j is constant then both of them are constants.
- A3 If $(z, j) \in E_j$ then for any $g \in \mathrm{SL}_2(C)$, $(gz, j) \in E_j$. Conversely, if for some j we have $(z_1, j), (z_2, j) \in E_j$ then $z_2 = g z_1$ for some $g \in \mathrm{SL}_2(C)$.
- A4 If $(z, j_1) \in E_j$ and $\Phi_N(j_1, j_2) = 0$ for some j_2 and some modular polynomial $\Phi_N(X, Y)$ then $(z, j_2) \in E_j$.
- AS If $(z_i, j_i) \in E_j,\ i = 1, \ldots, n$, with

$$\mathrm{td}_C C(\bar{z}, \bar{j}) \leq n,$$

then $\Phi_N(j_i, j_k) = 0$ for some N and some $1 \leq i < k \leq n$, or $j_i \in C$ for some i.

Definition 4.8. An *E_j-field* is a model of T_j^0. If K is an E_j-field, then a tuple $(\bar{z}, \bar{j}) \in K^{2n}$ is called an *E_j-point* if $(z_i, j_i) \in E_j(K)$ for each $i = 1, \ldots, n$. By abuse of notation, we let $E_j(K)$ denote the set of all E_j-points in K^{2n} for any natural number n.

It is easy to see that reducts of differential fields to the language $\mathfrak{L}_j$ are E_j-fields.

Let C be an algebraically closed field with $\mathrm{td}(C/\mathbb{Q}) = \aleph_0$ and let $\mathfrak{C}$ be the collection of all E_j-fields K with $C_K = C$. Note that C is an E_j-field with $E_j(C) = C^2$ and it is the smallest structure in $\mathfrak{C}$. From now on, by an E_j-field we understand a member of $\mathfrak{C}$.

Definition 4.9. For $B \in \mathfrak{C}$ and $X \subseteq B$ the $\mathfrak{C}$-*closure* of X inside B (or the $\mathfrak{C}$-substructure of B generated by X) is the structure[4]

$$\langle X \rangle_B := \bigcap_{A \in \mathfrak{C}: X \subseteq A \subseteq B} A.$$

A structure $A \in \mathfrak{C}$ is *finitely generated* if $A = \langle X \rangle_A$ for some finite $X \subseteq A$. The collection of all finitely generated structures from $\mathfrak{C}$ will be denoted by $\mathfrak{C}_{f.g.}$. Further, $A \subseteq_{f.g.} B$ means that A is a finitely generated structure of B.

Note that for some $X \subseteq A \in \mathfrak{C}$ we have $\langle X \rangle_A = C(X)^{\text{alg}}$ (with the induced structure from A) and $\mathfrak{C}_{f.g.}$ consists of those E_j-fields that have finite transcendence degree over C (which, in fact, are not finitely generated as $\mathfrak{L}_j$-structures).

Definition 4.10. For $A \subseteq B \in \mathfrak{C}_{f.g.}$ an E_j-*basis* of B over A is an E_j-point $\bar{b} = (\bar{z}, \bar{j})$ from B of maximal length satisfying the following conditions:

- j_i and j_k are modularly independent for all $i \neq k$,
- $(z_i, j_i) \notin A^2$ for each i.

We let $\sigma(B/A)$ be the length of $\bar{j}$ in an E_j-basis of B over A (equivalently, an E_j-basis of B over A has length $2\sigma(B/A)$). When $A = C$ we write $\sigma(B)$ for $\sigma(B/C)$. It is easy to see that for $A \subseteq B \in \mathfrak{C}_{f.g.}$ one has $\sigma(B/A) = \sigma(B) - \sigma(A)$. Further, for $A \in \mathfrak{C}_{f.g.}$ we define the *predimension* by

$$\delta(A) := \operatorname{td}_C A - \sigma(A).$$

Note that the Ax-Schanuel inequality implies that σ is finite for finitely generated structures. Moreover, for $A, B \subseteq G \in \mathfrak{C}_{f.g.}$ the inequality

$$\sigma(\langle A \cup B \rangle_G) \geq \sigma(A) + \sigma(B) - \sigma(A \cap B)$$

holds.[5] Hence δ is submodular, that is, for all A, B, G as above we have

$$\delta(\langle A \cup B \rangle_G) + \delta(A \cap B) \leq \delta(A) + \delta(B).$$

In terms of the predimension the Ax-Schanuel inequality states exactly that $\delta(A) \geq 0$ for all $A \in \mathfrak{C}_{f.g.}$ with equality holding if and only if $A = C$.

Definition 4.11. For $A, B \in \mathfrak{C}_{f.g.}$ the *relative predimension* of A over B is defined as $\delta(B/A) := \delta(AB) - \delta(A)$. This depends on a common extension of A and B, so we work in such a common extension without explicitly mentioning it. When $A \subseteq B$ we work in B and define $\delta(B/A) = \delta(B) - \delta(A)$.

Observe that for $A \subseteq B \in \mathfrak{C}_{f.g.}$

$$\delta(B/A) = \delta(B) - \delta(A) = \operatorname{td}(B/A) - \sigma(B/A).$$

In the next definition B is the ambient structure that we work in.

Definition 4.12. Let $A \subseteq B \in \mathfrak{C}$. We say A is *strong* (or *self-sufficient*) in B, denoted $A \leq B$, if for all $X \subseteq_{f.g.} B$ we have $\delta(X \cap A) \leq \delta(X)$. One also says B is a *strong extension* of A. An embedding $A \hookrightarrow B$ is strong if the image of A is strong in B.

For $M \in \mathfrak{C}$ and a finite set $\bar{a} \subseteq M$ we say $\bar{a}$ is strong in M if $\langle \bar{a} \rangle_M \leq M$.

Definition 4.13. For $B \in \mathfrak{C}$ and $X \subseteq B$ we define the *self-sufficient closure* (or *strong closure*) of X in B by

$$\lceil X \rceil_B := \bigcap_{A \in \mathfrak{C}: X \subseteq A \leq B} A.$$

[4]In the differential setting $\langle A \rangle$ stands for the differential subfield generated by a set A. The meaning of this notation will be clear from the context.

[5]Here and later we need to embed two structures into a third one in order to speak of the structure generated by those two structures (as well as their intersection). Often, when it does not cause any confusion, we will not mention the ambient structure explicitly .

A finite intersection of strong substructures is easily seen to be strong. An infinite intersection of algebraically closed fields of finite transcendence degree over C is actually a finite intersection, hence an infinite intersection of finitely generated strong substructures is strong as well. It follows from this that $\lceil X \rceil_B \leq B$. Note also that $\leq$ is transitive.

Lemma 4.14. *If $B \in \mathfrak{C}$ and $X \subseteq_{f.g.} B$ then*
- $\lceil X \rceil_B$ *is finitely generated, and*
- $\delta(\lceil X \rceil_B) = \min\{\delta(Y) : X \subseteq Y \subseteq_{f.g.} B\}$.

The predimension gives rise to a dimension (of a pregeometry) in the following way.

Definition 4.15. For $X \subseteq_{f.g.} B$ define

$$d_B(X) := \min\{\delta(Y) : X \subseteq Y \subseteq_{f.g.} B\} = \delta(\lceil X \rceil_B).$$

For a finite subset $X \subseteq_{fin} B$ set $d_B(X) := d_B(\langle X \rangle_B)$.

Define the operator $\mathrm{cl}_B : \mathfrak{P}(B) \to \mathfrak{P}(B)$ (the latter is the power set of B) by

$$\mathrm{cl}_B(X) = \{b \in B : d_B(b/X) = 0\}.$$

Then (B, cl_B) is a pregeometry and d_B is its dimension function.

Self-sufficient embeddings can be defined in terms of d. Indeed, if $A \subseteq B$ then $A \leq B$ if and only if for any $X \subseteq_{fin} A$ one has $d_A(X) = d_B(X)$.

Proposition 4.16. *The class $\mathfrak{C}$ is a* strong amalgamation class*, that is, the following conditions hold.*

C1 *Every $A \in \mathfrak{C}_{f.g.}$ has at most countably many finitely generated strong extensions up to isomorphism.*

C2 *$\mathfrak{C}$ is closed under unions of countable strong chains $A_0 \leq A_1 \leq \ldots$.*

SAP *$\mathfrak{C}_{f.g.}$ has the* strong amalgamation property*, that is, for all $A_0, A_1, A_2 \in \mathfrak{C}_{f.g.}$ with strong embeddings $A_0 \hookrightarrow A_i$, $i = 1, 2$, there is $B \in \mathfrak{C}_{f.g.}$ such that A_1 and A_2 can be strongly embedded into B over A, i.e. the corresponding diagram commutes.*

The following is a standard theorem.

Theorem 4.17 (Amalgamation theorem)**.** *There is a unique (up to isomorphism) countable structure $\mathcal{U} \in \mathfrak{C}$ with the following properties.*

U1 *$\mathcal{U}$ is* universal *with respect to strong embeddings, i.e. every countable $A \in \mathfrak{C}$ can be strongly embedded into $\mathcal{U}$.*

U2 *$\mathcal{U}$ is* saturated *with respect to strong embeddings, i.e. for every $A, B \in \mathfrak{C}_{f.g.}$ with strong embeddings $A \hookrightarrow \mathcal{U}$ and $A \hookrightarrow B$ there is a strong embedding of B into $\mathcal{U}$ over A.*[6]

Furthermore, $\mathcal{U}$ is homogeneous with respect to strong substructures, that is, any isomorphism between finitely generated strong substructures of $\mathcal{U}$ can be extended to an automorphism of $\mathcal{U}$.

This $\mathcal{U}$ is called the *(strong) Fraïssé limit* or the *Fraïssé-Hrushovski limit* of $\mathfrak{C}_{f.g.}$. It has a natural pregeometry associated with the predimension function as described above. It is also saturated and ω-stable of Morley rank ω (as $\mathfrak{L}_j$-reducts of differentially closed fields). For each n there is a unique generic n-type over any finite set which is realised by any n-tuple of dimension n.

Conjecture 4.18 ([Asl18a])**.** *Let $\mathcal{K}$ be the countable saturated differentially closed field. Then its reduct $\mathcal{K}_{E_j}$ is isomorphic to $\mathcal{U}$.*

In the terminology of [Asl18a] this conjecture states that the Ax-Schanuel inequality for the differential equation of the j-function is (strongly) adequate (which was also mentioned in the introduction). In the next section we will formulate an algebraic equivalent of this conjecture known as *Existential Closedness*.

[6] U2 is normally known as the *richness* property in the literature and it implies U1.

4.3. **Existential closedness.** Now we describe the Existential Closedness axiom scheme which, along with the above axioms, gives a complete axiomatisation of $\mathrm{Th}(\mathcal{U})$. We will also need it for our classification of strongly minimal sets in $\mathcal{K}_{E_j}$.

Definition 4.19. Let n be a positive integer, $k \leq n$ and $1 \leq i_1 < \ldots < i_k \leq n$. Denote $\bar{i} = (i_1, \ldots, i_k)$ and define the projection map $\mathrm{pr}_{\bar{i}} : K^n \to K^k$ by

$$\mathrm{pr}_{\bar{i}} : (x_1, \ldots, x_n) \mapsto (x_{i_1}, \ldots, x_{i_n}).$$

Further, define (by abuse of notation) $\mathrm{pr}_{\bar{i}} : K^{2n} \to K^{2k}$ by

$$\mathrm{pr}_{\bar{i}} : (\bar{x}, \bar{y}) \mapsto (\mathrm{pr}_{\bar{i}}\, \bar{x}, \mathrm{pr}_{\bar{i}}\, \bar{y}).$$

It will be clear from the context in which sense $\mathrm{pr}_{\bar{i}}$ should be understood.

Definition 4.20. Let K be an algebraically closed field. An irreducible algebraic variety $V \subseteq K^{2n}$ is *normal* if and only if for any $1 \leq k \leq n$ and any $1 \leq i_1 < \ldots < i_k \leq n$ we have $\dim \mathrm{pr}_{\bar{i}} V \geq k$. We say V is *strongly normal* if the strict inequality $\dim \mathrm{pr}_{\bar{i}} V > k$ holds.

Lemma 4.21. *If $A \leq B \in \mathfrak{C}_{f.g.}$ and $\bar{b}$ is an E_j-basis of B over A then $\mathrm{Loc}_A(\bar{b})$ is normal over A.*

Now consider the following properties of an E_j-field K.

- EC For each normal variety $V \subseteq K^{2n}$ the intersection $E_j(K) \cap V(K)$ is non-empty.
- SEC For each normal variety $V \subseteq K^{2n}$ defined over a finite tuple $\bar{a} \subseteq K$, the intersection $E_j(K) \cap V(K)$ contains a point Zariski generic in V over $\bar{a}$.
- NT $K \supsetneq C$.

EC, SEC and NT stand for Existential Closedness, Strong Existential Closedness and Non-Triviality respectively. Denote $T_j := T_j^0$+EC+NT (EC is first-order).

Proposition 4.22. *The strong Fraïssé limit $\mathcal{U}$ satisfies* SEC *(and hence* EC*).*

In fact, $T_j = \mathrm{Th}(\mathcal{U})$ and all $\aleph_0$-saturated models of T_j satisfy SEC.

Conjecture 4.23. *($\mathfrak{L}_j$-reducts of) differentially closed fields satisfy* EC.

This is equivalent to Conjecture 4.18. It states that if for a system of equations in $\mathcal{K}_{E_j}$ having a solution does not contradict Ax-Schanuel, then there is a solution. We will refer to both conjectures as the *Existential Closedness* conjecture or, briefly, EC conjecture.

4.4. **Types in $\mathcal{K}_{E_j}$.** *In this and the next sections we assume the EC conjecture.*

From now on we work in $\mathcal{K}_{E_j}$, the $\mathfrak{L}_j$-reduct of the countable saturated differentially closed field $\mathcal{K}$. Recall that by EC $\mathcal{K}_{E_j}$ is isomorphic to the strong Fraïssé limit $\mathcal{U}$. We will drop the subscript K from the the notations introduced in Section 4.2.

Lemma 4.24. *Let $\bar{a} \subseteq K$. If $(\bar{u}, \bar{v})$ is an E_j-basis of $\lceil \bar{a} \rceil$ then the latter is generated by $\bar{a}, \bar{u}, \bar{v}$.*

Proof. Let $A = C(\bar{a}, \bar{u}, \bar{v})^{\mathrm{alg}} \subseteq \lceil \bar{a} \rceil$. Then $\mathrm{td}(\lceil \bar{a} \rceil / C) \geq \mathrm{td}(A/C)$ and $\sigma(\lceil \bar{a} \rceil) = |\bar{v}| = \sigma(A)$. Hence $\delta(A) \leq \delta(\lceil \bar{a} \rceil)$ and so $\delta(A) = \delta(\lceil \bar{a} \rceil)$. Therefore $\mathrm{td}(\lceil \bar{a} \rceil / C) = \mathrm{td}(A/C)$ and $A = \lceil \bar{a} \rceil$. $\square$

Let $\bar{a} = (a_1, \ldots, a_m) \in K^m$ be a tuple with $d(\bar{a}) = k$ and $b \in K$ with $d(b/\bar{a}) = 0$, i.e. $b \in \mathrm{cl}(\bar{a})$. This means that $d(\bar{a}b) = k$. Pick an E_j-basis $(\bar{z}, \bar{j})$ of $B := \lceil \bar{a}, b \rceil$. By Lemma 4.24 $B = C(\bar{a}, b, \bar{z}, \bar{j})^{\mathrm{alg}}$. We claim that $b \in C(\bar{a}, \bar{z}, \bar{j})^{\mathrm{alg}}$. Indeed, if it is not true then

$$k = d(\bar{a}b) = \delta(B) > \delta(C(\bar{a}, \bar{z}, \bar{j})^{\mathrm{alg}}) \geq d(\bar{a}) = k.$$

Thus, $B = C(\bar{a}, \bar{z}, \bar{j})^{\mathrm{alg}}$. Let $p(\bar{a}, \bar{c}, \bar{z}, \bar{j}, b) = 0$ for some irreducible polynomial p and $\bar{c} \in C^s$. Denote $l := |\bar{j}| = \sigma(B)$ and $V := \mathrm{Loc}_{C(\bar{a})}(\bar{z}, \bar{j}) \subseteq K^{2l}$ and assume (by extending $\bar{c}$ if necessary) it is defined over $\bar{c}, \bar{a}$. In order to stress that V is defined over $\bar{a}, \bar{c}$, we denote it by $V_{\bar{a},\bar{c}}$ (it is a member of a parametric family of varieties). Notice that

$$\dim V_{\bar{a},\bar{c}} = \mathrm{td}(B/C) - \mathrm{td}(\bar{a}/C) = \delta(B) + \sigma(B) - \mathrm{td}(\bar{a}/C) = k + l - \mathrm{td}(\bar{a}/C). \quad (4.9)$$

Also, denote $W := \mathrm{Loc}_{\mathbb{Q}(\bar{a})}(\bar{c})$. For each proper Zariski closed subvariety $U \subsetneq W$, defined over $\mathbb{Q}(\bar{a})$, and each positive integer N consider the formulae

$$\xi_{U,N}(\bar{e}, \bar{u}, \bar{v}) := \left(\bar{e} \in C^{|\bar{c}|} \cap (W \setminus U) \wedge (\bar{u}, \bar{v}) \in V_{\bar{a},\bar{e}} \cap E_j^{\times} \wedge \bigwedge_{n=1}^{N} \bigwedge_{i \neq r} \Phi_n(v_i, v_r) \neq 0 \right),$$

$$\psi_{U,N}(\bar{e}, \bar{u}, \bar{v}, y) := \xi_{U,N}(\bar{e}, \bar{u}, \bar{v}) \wedge p(\bar{a}, \bar{e}, \bar{u}, \bar{v}, y) = 0,$$

$$\varphi_{U,N}(y) := \exists \bar{e}, \bar{u}, \bar{v}\ \psi_{U,N}(\bar{e}, \bar{u}, \bar{v}, y).$$

Observe that $\varphi_{U,N}$ is defined over $\bar{a}$ and $\varphi_{U,N}(b)$ holds in $\mathcal{K}_{E_j}$.

Proposition 4.25. *The formulae $\varphi_{U,N}$ axiomatise the type $\mathrm{tp}(b/\bar{a})$, that is, if for some $b' \in K$ the formula $\varphi_{U,N}(b')$ holds for each $U \subsetneq W$ and each $N > 0$ then $\mathrm{tp}(b/\bar{a}) = \mathrm{tp}(b'/\bar{a})$.*

Proof. Consider the type $q(\bar{e}, \bar{u}, \bar{v})$ over $\bar{a}, b'$ consisting of all formulae $\psi_{U,N}(\bar{e}, \bar{u}, \bar{v}, b')$ for all $U \subsetneq V$ and $N > 0$. Then q is finitely satisfiable and hence there is a realisation of q in $\mathcal{K}_{E_j}$ which we denote by $\bar{c}', \bar{z}', \bar{j}'$.

Observe that $\bar{c}'$ is generic in W over $\bar{a}$. Hence $\bar{c}$ and $\bar{c}'$ have the same algebraic type over $\bar{a}$. In particular, $\dim V_{\bar{a},\bar{c}} = \dim V_{\bar{a},\bar{c}'}$. Further, $j_1', \ldots, j_l'$ are pairwise modularly independent. So if $B' := C(\bar{a}, \bar{z}', \bar{j}')^{\mathrm{alg}}$ then $\sigma(B') \geq l$. On the other hand

$$\mathrm{td}(B'/C) = \mathrm{td}(\bar{a}/C) + \mathrm{td}(\bar{z}', \bar{j}'/C(\bar{a})) \leq \mathrm{td}(\bar{a}/C) + \dim V_{\bar{a},\bar{c}'} = k + l, \tag{4.10}$$

where the last equality follows from (4.9). Therefore $\delta(B') \leq (k + l) - l = k$. However, B' contains $\bar{a}$ and since $d(\bar{a}) = k$, $\delta(B')$ cannot be smaller than k. Thus, $\delta(B') = k$ and $\sigma(B') = l$ and the inequality in (4.10) must be an equality, i.e. $\mathrm{td}(\bar{z}', \bar{j}'/C(\bar{a})) = \dim V_{\bar{a},\bar{c}'}$. This means that $\bar{z}', \bar{j}'$ is generic in $V_{\bar{a},\bar{c}'}$ over $C(\bar{a})$. Therefore, there is a field isomorphism $\pi : B \to B'$ which fixes $\bar{a}$ pointwise, fixes C setwise, sends $\bar{c}$ to $\bar{c}'$ and sends $(\bar{z}, \bar{j})$ to $(\bar{z}', \bar{j}')$. Since $\sigma(B') = l$, the tuple $(\bar{z}', \bar{j}')$ is an E_j-basis of B' and π is an isomorphism of B and B' as E_j-fields.

Finally, as $p(\bar{a}, \bar{c}, \bar{z}, \bar{j}, b) = 0$ and $p(\bar{a}, \bar{c}', \bar{z}', \bar{j}', b') = 0$, we could have chosen π so that $\pi(b) = \pi(b')$. Now both B and B' are strong in $\mathcal{K}_{E_j}$ and the latter is homogeneous with respect to strong substructures, hence π can be extended to an automorphism of $\mathcal{K}_{E_j}$. This shows that b and b' have the same type over $\bar{a}$. $\square$

Remark 4.26. In general, all types in $\mathcal{K}_{E_j}$ are determined by formulas of the above form and their negations. In particular, every formula is equivalent to a Boolean combination of existential formulas in $\mathcal{K}_{E_j}$ and hence its theory is nearly model complete.

Theorem 4.27. *If $\bar{a} \leq K$ then $\mathrm{acl}(C(\bar{a})) = C(\bar{a})^{\mathrm{alg}}$.*

Proof. It suffices to prove that for $\bar{a} \leq K$ we have $\mathrm{acl}(\bar{a}) \subseteq C(\bar{a})^{\mathrm{alg}}$. Assume $b \in \mathrm{acl}(\bar{a})$. Then $d(b/\bar{a}) = 0$ and $\mathrm{tp}(b/\bar{a})$ is determined by existential formulas $\varphi_{U,N}(y)$. Since $b \in \mathrm{acl}(\bar{a})$, some formula $\varphi_{U,N}(y) \in \mathrm{tp}(b/\bar{a})$ has finitely many realisations in $\mathcal{K}_{E_j}$.

We use the above notation. The point $(\bar{z}, \bar{j}) \in V_{\bar{a},\bar{c}}$ is generic over $\bar{a}, \bar{c}$. Observe that $(\bar{z}, \bar{j})$ must contain an E_j-basis of $A = C(\bar{a})^{\mathrm{alg}}$. Denote it by $(\bar{z}_{\bar{a}}, \bar{j}_{\bar{a}})$ and $(\bar{z}_0, \bar{j}_0) := (\bar{z}, \bar{j}) \setminus (\bar{z}_{\bar{a}}, \bar{j}_{\bar{a}})$, i.e. $(\bar{z}_0, \bar{j}_0)$ consists of all coordinates (z_i, j_i) of $(\bar{z}, \bar{j})$ for which $(z_i, j_i) \notin A^2$. In other words, $(\bar{z}_0, \bar{j}_0)$ is an E_j-basis of $B = \lceil A(b) \rceil$ over A. Let W be an irreducible component over $L := \mathbb{Q}(\bar{a}, \bar{c}, \bar{z}_{\bar{a}}, \bar{j}_{\bar{a}})^{\mathrm{alg}}$ of the fibre of $V_{\bar{a},\bar{c}}$ above $(\bar{z}_{\bar{a}}, \bar{j}_{\bar{a}})$ containing $(\bar{z}_0, \bar{j}_0)$. Then it is defined over L and $(\bar{z}_0, \bar{j}_0)$ is generic in W over L.

Since $\varphi_{U,N}(b)$ holds, in particular we have $p(\bar{a}, \bar{c}, \bar{z}, \bar{j}, b) = 0$. Assume

$$p(\bar{a}, \bar{c}, \bar{z}, \bar{j}, Y) = Y^n + s_{n-1}(\bar{z}_0, \bar{j}_0) Y^{n-1} + \cdots + s_0(\bar{z}_0, \bar{j}_0)$$

where each $s_i(\bar{X}_1, \bar{X}_2)$ is a rational functions over L. If for all i $s_i(\bar{z}_0, \bar{j}_0) \in L$ then $b \in L \subseteq A$. Otherwise assume without loss of generality that $s_0(\bar{z}_0, \bar{j}_0) \notin L$.

Since $A \leq B$, by Lemma 4.21 W is normal. By SEC there is a point $(\bar{z}_1, \bar{j}_1) \in W(K) \cap E_j(K)$ generic over $L(\bar{z}_0, \bar{j}_0)$. If $s_0(\bar{z}_1, \bar{j}_1) = s_0(\bar{z}_0, \bar{j}_0)$ then the function $s_0(\bar{X}_1, \bar{X}_2)$ is constant on W. On the other hand, W is defined over L, so the constant value of $s_0(\bar{X}_1, \bar{X}_2)$ must belong to L.

This is a contradiction, hence $s_0(\bar{z}_1, \bar{j}_1) \neq s_0(\bar{z}_0, \bar{j}_0)$. Now pick a generic point $(\bar{z}_2, \bar{j}_2)$ in $W(K) \cap E_j(K)$ over $L(\bar{z}_0, \bar{j}_0, \bar{z}_1, \bar{j}_1)$. By the above argument the elements $s_0(\bar{z}_0, \bar{j}_0), s_0(\bar{z}_1, \bar{j}_1), s_0(\bar{z}_2, \bar{j}_2)$ are pairwise distinct. Iterating this process we will construct a sequence $(\bar{z}_i, \bar{j}_i), i = 0, 1, 2, \ldots$ such that for each i

$$\mathcal{K}_{E_j} \models \xi_{U,N}(\bar{c}, \bar{z}_{\bar{a}}, \bar{z}_i, \bar{j}_{\bar{a}}, \bar{j}_i)$$

and $s_0(\bar{z}_i, \bar{j}_i), i = 0, 1, 2, \ldots$ are pairwise distinct. This shows that the formula $\varphi_{U,N}(y)$ has infinitely many realisations (for there are only finitely many monic polynomials of a given degree the roots of which belong to a finite set of elements). This is a contradiction. $\square$

Corollary 4.28. *For any $\bar{a} \subseteq K$ we have* $\operatorname{acl}(\bar{a}) \subseteq \lceil \bar{a} \rceil$.

4.5. **Classification of strongly minimal sets in $\mathcal{K}_{E_j}$.** *Recall that we assume the EC conjecture.*

Theorem 4.29. *Let $S \subseteq K$ be a strongly minimal set. Then either S is geometrically trivial or $S \not\perp C$.*

Proof. Assume S is defined over $\bar{a}$ and $S \perp C$. Denote $A := C(\bar{a})^{\mathrm{alg}}$. Pick $b \in S \setminus \operatorname{acl}(A)$ (if such an element does not exist then $S \not\perp C$). Then clearly $d(b/A) = 0$. Denote $B' := \lceil Ab \rceil$ and let $z_b \in K$ be such that $E_j(z_b, b)$ holds. Now if $B = B'(z_b)^{\mathrm{alg}}$ (with the induced structure from $\mathcal{K}_{E_j}$) then $\delta(B) = \delta(B') = d(A)$ as $d(b/A) = 0$. Hence $B \leq K$. Choose a maximal E_j-field A' with $A \subseteq A' \subseteq B$ such that $b \notin \operatorname{acl}(A')$. Since strong minimality of a set and the nature of the geometry of a strongly minimal set do not depend on the choice of the set of parameters over which the set if defined, we may extend A and assume $A' = A$. This means that if $e \in B \setminus A$ then $b \in \operatorname{acl}(Ae)$. In particular, $\operatorname{acl}(A) = A$.

Let $(\bar{z}, \bar{j}) \in B^{2l}$ be an E_j-basis of B with $j_l = b$. Further, extending $\bar{a}$ we may assume that $V := \operatorname{Loc}_A(\bar{z}, \bar{j}) \subseteq K^{2l}$ is defined over $\bar{a}$. Then $\operatorname{tp}(b/A)$ is determined by the formulae

$$\chi_N(y) := \exists \bar{u}, \bar{v} \left((\bar{u}, \bar{v}) \in V \cap E_j^{\times} \wedge y = v_l \wedge \bigwedge_{n=1}^{N} \bigwedge_{i \neq r} \Phi_n(v_i, v_r) \neq 0 \right).$$

Now pick pairwise acl-independent elements $b_1, \ldots, b_t \in S \setminus A$. We will show that $b_t \notin \operatorname{acl}(Ab_1 \ldots b_{t-1})$. Since S is strongly minimal, $\operatorname{tp}(b_i/A) = \operatorname{tp}(b/A)$ for all i. By saturatedness of $\mathcal{K}_{E_j}$ for each i there is $(\bar{z}^i, \bar{j}^i) \in V \cap E_j^{\times}$ such that $\bar{j}^i$ is pairwise modularly independent and $j_l^i = b_i$. Denote $B_i = A(\bar{z}^i, \bar{j}^i)^{\mathrm{alg}}$.

It is clear that $\dim V = \operatorname{td}_A(\bar{z}, \bar{j}) = \operatorname{td}(B/A) = \delta(B/A) + \sigma(B/A)$. Therefore

$$\begin{aligned} \delta(B_i) = \operatorname{td}(B_i/C) - \sigma(B_i) \leq \dim V + \operatorname{td}(A/C) - l = \\ \delta(B/A) + \sigma(B/A) + \delta(A) + \sigma(A) - l = \delta(B) = d(A), \end{aligned}$$

and so $B_i \leq K$ and $(\bar{z}^i, \bar{j}^i)$ is an E_j-basis of B_i. We can conclude now that $\lceil Ab_i \rceil \subseteq B_i$, hence $\operatorname{acl}(Ab_i) \subseteq B_i$. Moreover, as in the previous section there is an automorphism of $\mathcal{K}_{E_j}$ over A that maps B onto B_i (and maps $(\bar{z}, \bar{j})$ to $(\bar{z}^i, \bar{j}^i)$). In particular, for every $e \in B_i \setminus A$ we have $b_i \in \operatorname{acl}(Ae)$.

We claim that j_r^i and j_k^m are modularly independent unless $(i, r) = (m, k)$ or $j_r^i, j_k^m \in A$. Assume for contradiction that for some $i \neq m$ the elements j_r^i and j_k^m are modularly dependent and $j_r^i \notin A$. Then $b_i \in \operatorname{acl}(Aj_r^i) = \operatorname{acl}(Aj_k^m) \subseteq B_m$. Hence $b_m \in \operatorname{acl}(Ab_i)$ which is a contradiction, for we assumed b_i's are pairwise acl-independent. This shows in particular that (when $t \geq 2$) $A \leq K$ as otherwise we would have $b \in \lceil A \rceil$ and $S \subseteq \operatorname{acl}(Ab)$ in which case $S \not\perp C$.

Now let $\tilde{B}_k := B_1 \ldots B_k$ be the E_j-subfield of $\mathcal{K}_{E_j}$ generated by $B_1, \ldots, B_k$ where $k \leq t$. The above argument shows that

$$\sigma(\tilde{B}_k/A) = \sum_{i=1}^{k} \sigma(B_i/A) = k \cdot \sigma(B/A).$$

By submodularity of δ we have

$$\delta(\tilde{B}_k) \leq \delta(\tilde{B}_{k-1}) + \delta(B_k) - \delta(\tilde{B}_{k-1} \cap B_k)$$

for each k. Since $\delta(\tilde{B}_{k-1} \cap B_k) \geq d(A)$, we can show by induction that $\delta(\tilde{B}_k) = d(A)$ and $\tilde{B}_k \leq K$. Thus,

$$\mathrm{td}_C(\tilde{B}_k) = \delta(\tilde{B}_k) + \sigma(\tilde{B}_k) = d(A) + \sigma(\tilde{B}_k). \tag{4.11}$$

On the other hand, using submodularity of td and $-\sigma$ we get by induction

$$\begin{aligned} \mathrm{td}_C(\tilde{B}_k) \leq \mathrm{td}_C(\tilde{B}_{k-1}) + \mathrm{td}_C(B_k) - \mathrm{td}_C(\tilde{B}_{k-1} \cap B_k) = \\ d(A) + \sigma(\tilde{B}_{k-1}) + d(A) + \sigma(B_k) - \delta(\tilde{B}_{k-1} \cap B_k) - \sigma(\tilde{B}_{k-1} \cap B_k) \\ \leq d(A) + \sigma(\tilde{B}_k), \end{aligned}$$

where $\delta(\tilde{B}_{k-1} \cap B_k) \geq d(A)$ for $A \subseteq \tilde{B}_{k-1} \cap B_k$. In fact we must have equalities everywhere in the above inequality due to (4.11). In particular,

$$\sigma((\tilde{B}_{k-1} \cap B_k)/A) = \sigma(\tilde{B}_{k-1}/A) + \sigma(B_k/A) - \sigma(\tilde{B}_k/A) = 0.$$

So

$$\mathrm{td}((\tilde{B}_{k-1} \cap B_k)/A) = \delta((\tilde{B}_{k-1} \cap B_k)/A) + \sigma((\tilde{B}_{k-1} \cap B_k)/A) = 0.$$

This implies that $\tilde{B}_{k-1} \cap B_k = A$. In particular, $b_t \notin \tilde{B}_{t-1}$. On the other hand, $\mathrm{acl}(Ab_1 \dots b_{t-1}) \subseteq \lceil Ab_1 \dots b_{t-1} \rceil \subseteq \tilde{B}_{t-1}$. Thus, $b_t \notin \mathrm{acl}(Ab_1 \dots b_{t-1})$ as required. □

We can also prove that some sets are strongly minimal. Let $A := C(\bar{a})^{\mathrm{alg}} \leq K$. Assume $V \subseteq K^2$ is an algebraic curve defined over A, i.e. $\dim V = 1$. Consider the formula

$$\chi(y) := \exists \bar{u}, \bar{v} \left((\bar{u}, \bar{v}) \in V \cap E_j^{\times} \wedge p(\bar{a}, \bar{u}, \bar{v}, y) = 0 \right),$$

where p is some irreducible algebraic polynomial.

Proposition 4.30. *If $S := \chi(\mathcal{K}_{E_j})$ is infinite then S is strongly minimal.*

Proof. We need to show that over any set of parameters all non-algebraic elements in S realise the same type. By the stable embedding property we may choose all extra parameters from the set S itself. Assume $e, e', b_1, \dots, b_t \in S$ with $e, e' \notin A(\bar{b})^{\mathrm{alg}}$. We will show that $\mathrm{tp}(e/A(\bar{b})) = \mathrm{tp}(e'/A(\bar{b}))$.

Choose existential witnesses $(z, j), (z', j'), (z_i, j_i) \in V(K) \cap E_j^{\times}(K)$ for $\chi(e)$, $\chi(e')$ and $\chi(b_i)$ respectively. Since $e \notin A(\bar{b})^{\mathrm{alg}}$ and $p(\bar{a}, z, j, e) = 0$ and $\dim V = 1$, the point (z, j) is generic in V over $A(\bar{b})$. Similarly (z', j') is generic in V. So (z, j) and (z', j') have the same algebraic type over $A(\bar{b})$. On the other hand, $\delta(\bar{b}/A) \leq 0$, therefore $\delta(\bar{b}/A) = 0$. Thus $\delta(e/A\bar{b}) = \delta(e'/A\bar{b})$ and (z, j) and (z', j') form E_j-bases of $A(\bar{b}, e)^{\mathrm{alg}}$ and $A(\bar{b}, e')^{\mathrm{alg}}$ over $A(\bar{b})^{\mathrm{alg}}$ respectively. Hence, as in the proof of Proposition 4.25, e can be mapped to e' by an automorphism of $\mathcal{K}_{E_j}$ over $A(\bar{b})$. □

Remark 4.31. When A is not strong in K we may actually work over $\lceil A \rceil$ since strong minimality of a set does not depend on the choice of the set of parameters over which the set if defined. Hence the assumption $A \leq K$ does not restrict generality.

Appendix A. On strong minimality

In this appendix we give some preliminaries on strongly minimal sets. For a detailed account of strongly minimal sets and geometric stability theory in general we refer the reader to [Pil96].

Algebraic closure defines a pregeometry on a strongly minimal set. More precisely, if X is a strongly minimal set in a structure $\mathcal{M}$ defined over $A \subseteq M$ then the operator

$$\mathrm{cl} : Y \mapsto \mathrm{acl}(AY) \cap X, \text{ for } Y \subseteq X,$$

is a pregeometry.

Definition A.1. Let $\mathcal{M}$ be a structure and $X \subseteq M$ be a strongly minimal set defined over a finite set $A \subseteq M$.

- We say X is *geometrically trivial* (over A) if whenever $Y \subseteq X$ and $z \in \operatorname{acl}(AY) \cap X$ then $z \in \operatorname{acl}(Ay)$ for some $y \in Y$. In other words, geometric triviality means that the closure of a set is equal to the union of closures of singletons.
- X is called *strictly disintegrated* (over A) if any distinct elements $x_1, \ldots, x_n \in X$ are independent (over A).
- X is called $\aleph_0$-*categorical* (over A) if it realises only finitely many 1-types over AY for any finite $Y \subseteq X$. This is equivalent to saying that $\operatorname{acl}(AY) \cap X$ is finite for any finite $Y \subseteq X$.

Note that strict disintegratedness implies $\aleph_0$-categoricity and geometric triviality.

Theorem A.2. *Let $\mathcal{M}$ be a model of an ω-stable theory and $X \subseteq M$ be as above. If X is geometrically trivial over A then it is geometrically trivial over any superset $B \supseteq A$.*

Proof. By expanding the language with constant symbols for elements of A we can assume that X is $\emptyset$-definable. Also we can assume $B = \{b_1, \ldots, b_n\}$ is finite. Let $z \in \operatorname{acl}(BY)$ for some finite $Y \subseteq X$. By stability $\operatorname{tp}(\bar{b}/X)$ is definable over a finite $C \subseteq X$ and we may assume that $C \subseteq \operatorname{acl}(B) \cap X$. Therefore $z \in \operatorname{acl}(CY)$. By geometric triviality of X (over $\emptyset$) we have $z \in \operatorname{acl}(c)$ for some $c \in C$ or $z \in \operatorname{acl}(y)$ for some $y \in Y$. This shows geometric triviality of X over B. $\square$

As we saw in the proof all definable subsets of X^m over B are definable over $\operatorname{acl}(B) \cap X$ (which means that X is *stably embedded* into M). The same argument shows that $\aleph_0$-categoricity does not depend on parameters (cf. [NP17, Lemma 2.20]). Of course this is not true for strict disintegratedness but a weaker property is preserved. Namely, if X is strictly disintegrated over A then any distinct non-algebraic elements over B are independent over B.

Definition A.3. Two definable sets X and Y are called *orthogonal*, written $X \perp Y$, iff any two elements $x \in X$ and $y \in Y$ are (forking) independent over any set of parameters over which X and Y are defined.

The following gives a simpler characterisation of orthogonality for strongly minimal sets.

Lemma A.4. *Two strongly minimal sets X and Y are non-orthogonal iff for some finite parameter set A we have $Y \subseteq \operatorname{acl}(A \cup X)$.*

Non-orthogonality means that the given sets are "similar". It is an equivalence relation for strongly minimal sets.

Acknowledgements. The first part of this work was done while I was a DPhil student at the University of Oxford under the supervision of Boris Zilber and Jonathan Pila. I would like to thank them for valuable suggestions and comments.

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