Two-sided bounds for L_p-norms of combinations of products of independent random variables

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Abstract

We show that for every positive p, the L_p-norm of linear combinations (with scalar or vector coefficients) of products of i.i.d. nonnegative random variables with the p-norm one is comparable to the l_p-norm of the coefficients and the constants are explicit. As a result the same holds for linear combinations of Riesz products.

We also establish the upper and lower bounds of the L_p-moments of partial sums of perpetuities.

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1 Introduction and Main Results

Let $X, X_1, X_2, \ldots$ be i.i.d. nondegenerate nonnegative r.v.'s with finite mean. Define

$$R_0 := 1 \quad \text{and} \quad R_i := \prod_{j=1}^{i} X_j \quad \text{for } i = 1, 2, \ldots. \tag{1}$$

Then obviously for any vectors $v_0, v_1, \ldots, v_n$ in a normed space $(F, \| \ \|)$, $\mathbb{E}\| \sum_{i=0}^n v_i R_i \| \leq \sum_{i=0}^n \|v_i\| \mathbb{E} R_i$. In [17] it was shown that the opposite inequality holds, i.e.

$$\mathbb{E} \left\| \sum_{i=0}^n v_i R_i \right\| \geq c_X \sum_{i=0}^n \|v_i\| \mathbb{E} R_i,$$

where c_X is a constant, which depends only on the distribution of X.

In this paper we present similar estimates for L_p-norms. Our main result is the following.

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Theorem 1. *Let $p > 0$ and $X, X_1, X_2, \dots$ be i.i.d. nondegenerate nonnegative r.v.'s such that $\mathbb{E}X^p < \infty$ and let R_i be defined by* (1)*. Then there exist constants $0 < c_{p,X} \le C_{p,X} < \infty$ which depend only on p and the distribution of X such that for any vectors $v_0, v_1, \dots, v_n$ in a normed space $(F, \| \, \|)$,*

$$c_{p,X} \sum_{i=0}^{n} \|v_i\|^p \mathbb{E}R_i^p \le \mathbb{E}\left\|\sum_{i=0}^{n} v_i R_i\right\|^p \le C_{p,X} \sum_{i=0}^{n} \|v_i\|^p \mathbb{E}R_i^p.$$

In fact we prove a more general result that does not require the identical distribution assumption. Namely, suppose that

$$X_1, X_2, \dots \text{ are independent, nonnegative r.v.'s such that } \mathbb{E}X_i^p < \infty. \tag{2}$$

Further assumptions depend on whether $p \le 1$. For $p \in (0,1]$ we assume that

$$\exists_{\lambda<1}\ \forall_i\ \mathbb{E}X_i^{p/2} \le \lambda(\mathbb{E}X_i^p)^{1/2} \tag{3}$$

and

$$\exists_{0<\delta<1, A>1}\ \forall_i\ \mathbb{E}(X_i^p - \mathbb{E}X_i^p)\mathbb{1}_{\{\mathbb{E}X_i^p \le X_i^p \le A\mathbb{E}X_i^p\}} \ge \delta \mathbb{E}X_i^p. \tag{4}$$

Theorem 2. *Let $0 < p \le 1$ and $X_1, X_2, \dots$ satisfy assumptions* (2)*,* (3) *and* (4)*. Then for any vectors $v_0, v_1, \dots, v_n$ in a normed space $(F, \| \, \|)$ we have*

$$c(p,\lambda,\delta,A) \sum_{i=0}^{n} \|v_i\|^p \mathbb{E}R_i^p \le \mathbb{E}\left\|\sum_{i=0}^{n} v_i R_i\right\|^p \le \sum_{i=0}^{n} \|v_i\|^p \mathbb{E}R_i^p,$$

where $c(p,\lambda,\delta,A)$ is a constant which depends only on p, λ, δ and A.

For $p > 1$ to obtain the lower bound we assume that

$$\exists_{\mu>0, A<\infty}\ \forall_i\ \mathbb{E}|X_i - \mathbb{E}X_i| \ge \mu(\mathbb{E}X_i^p)^{1/p} \text{ and } \mathbb{E}|X_i - \mathbb{E}X_i|\mathbb{1}_{\{X_i > A(\mathbb{E}X_i^p)^{1/p}\}} \le \frac{1}{4}\mu(\mathbb{E}X_i^p)^{1/p} \tag{5}$$

and

$$\exists_{q>\max\{p-1,1\}}\ \exists_{\lambda<1}\ \forall_i\ (\mathbb{E}X_i^q)^{1/q} \le \lambda(\mathbb{E}X_i^p)^{1/p}. \tag{6}$$

For the upper bound we need the condition

$$\forall_{k=1,2,\dots,\lceil p\rceil-1}\ \exists_{\lambda_k<1}\ \forall_i\ (\mathbb{E}X_i^{p-k})^{1/(p-k)} \le \lambda_k(\mathbb{E}X_i^{p-k+1})^{1/(p-k+1)}. \tag{7}$$

Theorem 3. *Let $p > 1$ and $X_1, X_2, \dots$ satisfy assumptions* (2)*,* (5)*,* (6) *and* (7)*. Then for any vectors $v_0, v_1, \dots, v_n$ in a normed space $(F, \| \, \|)$ we have*

$$c(p,\mu,A,q,\lambda) \sum_{i=0}^{n} \|v_i\|^p \mathbb{E}R_i^p \le \mathbb{E}\left\|\sum_{i=0}^{n} v_i R_i\right\|^p \le C(p,\lambda_1,\dots,\lambda_{\lceil p\rceil-1}) \sum_{i=0}^{n} \|v_i\|^p \mathbb{E}R_i^p,$$

where $c(p,\mu,A,q,\lambda)$ is a positive constant which depends only on p, μ, A, q and λ and $C(p,\lambda_1,\dots,\lambda_{\lceil p\rceil-1})$ is a constant which depends only on $p, \lambda_1, \dots, \lambda_{\lceil p\rceil-1}$.

Remark. Proofs presented below show that Theorem 2 holds with

$$c(p,\lambda,\delta,A)=\frac{\delta^3}{16k},\ \text{where } k \text{ is an integer such that } k\lambda^{2k-2}\leq \frac{\delta^3(1-\lambda)^2}{2^{12}A}.$$

In Theorem 3 we can take

$$C(p,\lambda_1,\ldots,\lambda_{\lceil p\rceil-1})=2^{\frac{p(p+1)}{2}}\prod_{1\leq j\leq \lceil p\rceil-1}\frac{1}{1-\lambda_j^{p-j}}$$

and

$$c(p,\mu,A,q,\lambda)=\frac{\mu^{3p}}{8k\cdot 2^{10p}\cdot 3^p},\ \text{where } k \text{ is an integer such that } k\lambda^{pk}\leq \frac{(1-\lambda)\mu^{3p}}{8C_0\cdot 2^{10p}\cdot 3^p},$$

$$C_0=(1-\lambda)^{1-p}\left(\frac{2A}{3\lambda}\right)^p\left(\frac{2p}{(q+1-p)\ln 2}\right)^{\frac{p}{q}}48^{\frac{2p^2}{\min\{p-1,1\}}}.$$

Theorem 1 yields by conditioning a similar result for products of symmetric r.v.'s.

Corollary 4. *Let $p>0$ and $X,X_1,X_2,\ldots$ be an i.i.d. sequence of symmetric r.v.'s such that $\mathbb{E}|X|^p<\infty$ and $\mathbb{P}(|X|=t)<1$ for all t. Then there exist constants $0<c_{p,X}\leq C_{p,X}<\infty$ which depend only on p and the distribution of X such that for any vectors $v_0,v_1,\ldots,v_n$ in a normed space $(F,\|\ \|)$,*

$$c_{p,X}\sum_{i=0}^n\|v_i\|^p\mathbb{E}|R_i|^p\leq \mathbb{E}\left\|\sum_{i=0}^n v_iR_i\right\|^p\leq C_{p,X}\sum_{i=0}^n\|v_i\|^p\mathbb{E}|R_i|^p.$$

Proof. Let (ε_i) be a sequence of independent symmetric ±1 r.v.'s, independent of (X_i). Then

$$\mathbb{E}\left\|\sum_{i=0}^n v_iR_i\right\|^p=\mathbb{E}\left\|\sum_{i=0}^n v_i\prod_{k=1}^i\varepsilon_k\prod_{k=1}^i|X_k|\right\|^p$$

and it is enough to use Theorem 1 for variables $(|X_i|)$. □

Another consequence of Theorem 1 is an estimate for L_p-norms of linear combinations of Riesz products. Let $\mathbb{T}=\mathbb{R}/2\pi\mathbb{Z}$ be the one dimensional torus and m be the normalized Haar measure on $\mathbb{T}$. Riesz products are defined on $\mathbb{T}$ by the formula

$$\bar{R}_i(t)=\prod_{j=1}^i(1+\cos(n_jt)),\quad i=1,2,\ldots,\tag{8}$$

where $(n_k)_{k\geq1}$ is a lacunary increasing sequence of positive integers.

It is well known that if coefficients n_k grow sufficiently fast then $\|\sum_{i=0}^n a_i\bar{R}_i\|_{L_p(\mathbb{T})}\sim(\mathbb{E}|\sum_{i=0}^n a_iR_i|^p)^{1/p}$ for $p\geq1$, where R_i are products of independent random variables distributed as $\bar{R}_1$. Together with Theorem 1 this gives an estimate for $\|\sum_{i=0}^n a_i\bar{R}_i\|_{L_p(\mathbb{T})}$. Here is the more quantitative result.

Corollary 5. *Suppose that $(n_k)_{k\geq 1}$ is an increasing sequence of positive integers such that $n_{k+1}/n_k \geq 3$ and $\sum_{k=1}^{\infty} \frac{n_k}{n_{k+1}} < \infty$. Then for any coefficients $a_0, a_1, \ldots, a_n \in \mathbb{R}$ and $p \geq 1$,*

$$c_p \sum_{i=0}^{n} |a_i|^p \int_{\mathbb{T}} |\bar{R}_i(t)|^p \,\mathrm{d}m(t) \leq \int_{\mathbb{T}} \left| \sum_{i=0}^{n} a_i \bar{R}_i(t) \right|^p \,\mathrm{d}m(t) \leq C_p \sum_{i=0}^{n} |a_i|^p \int_{\mathbb{T}} |\bar{R}_i(t)|^p \,\mathrm{d}m(t), \tag{9}$$

where $0 < c_p \leq C_p < \infty$ are constants depending only on p and the sequence (n_k).

Proof. Let $X_1, X_2, \ldots$ be independent random variables distributed as $1 + \cos(Y)$, where Y is uniformly distributed on $[0, 2\pi]$ and R_i be as in (1). By the result of Y. Meyer [18], $\| \sum_{i=0}^{n} a_i \bar{R}_i \|_{L_p} \sim (\mathbb{E} | \sum_{i=0}^{n} a_i R_i|^p)^{1/p}$ (in particular also $\|\bar{R}_i\|_{L_p} \sim (\mathbb{E} R_i^p)^{1/p}$) and the estimate follows by Theorem 1. □

Theorem 1 has also an immediate application to the stationary solution S of the random difference equation

$$S = XS + B, \tag{10}$$

where the equality is meant in law, (X, B) is a random variable with values in $[0, \infty) \times \mathbb{R}^d$ independent of S such that for some $p > 0$,

$$\mathbb{E}X^p = 1, \quad \mathbb{E}\|B\|^p < \infty \quad \text{and} \quad \mathbb{P}(X = 1) < 1. \tag{GK1}$$

Over the last 40 years equation (10) and its various modifications have attracted a lot of attention [1, 2, 3, 5, 8, 11, 12, 13, 14, 15, 16, 19, 20]. It has a rather wide spectrum of applications including random walks in random environment, branching processes, fractals, finance, insurance, telecommunications, various physical and biological models. In particular, the tail behaviour of S is of interest.

It is well known that in law

$$S = \sum_{i=1}^{\infty} R_{i-1} B_i,$$

where $R_{i-1} = X_1 \cdots X_{i-1}, R_0 = 1$ and (X_i, B_i) is an i.i.d sequence of r.v.'s with the same distribution as (X, B). Under the additional assumption that

$$\log X \text{ conditioned on } \{X \neq 0\} \text{ is non lattice and } \mathbb{E}X^p \log^+ X < \infty, \tag{GK2}$$

S has a heavy tail behaviour, i.e. the limit

$$\lim_{t\to\infty} t^p \mathbb{P}(\|S\| > t) = c_\infty(X, B)$$

exists and $c_\infty(X, B)$ is strictly positive provided that $\mathbb{P}(Xv + B = v) < 1$ for every $v \in \mathbb{R}^d$. If $\mathbb{P}(Xv + B = v) = 1$ then $S_n = v - R_{n-1}v \to v = S$. Assumptions (GK1), (GK2) together with $\mathbb{P}(Xv + B = v) < 1$ will be later on referred to as the *Goldie-Kesten conditions*. Let

$$S_n = \sum_{i=1}^{n} R_{i-1} B_i.$$

It turns out that the sequence $\mathbb{E}\|S_n\|^p$ is closely related to $c_\infty(X, B)$. Recently, it has been proved in [6] that under the *Goldie-Kesten conditions* plus a little bit stronger moment assumption $\mathbb{E}(X^{p+\varepsilon} + \|B\|^{p+\varepsilon}) < \infty$ for some $\varepsilon > 0$, we have

$$\lim_{n\to\infty} \frac{1}{np\rho} \mathbb{E}\|S_n\|^p = c_\infty(X, B) > 0,$$

where $\rho := \mathbb{E}X^p \log X$.

Now suppose that X, B are independent. Then Theorem 1 implies that for every n

$$c_{p,X} \mathbb{E}\|B\|^p \leq \frac{1}{n} \mathbb{E}\|S_n\|^p \leq C_{p,X} \mathbb{E}\|B\|^p, \tag{11}$$

which gives uniform bounds on the Goldie constant $c_\infty(X, B)$ depending only on the law of X and $\mathbb{E}\|B\|^p$ and independent of the dimension. Moreover, in some particular cases when constants $\lambda, \delta, \mu, q, \lambda_k$ in (3)–(7) can be estimated more carefully, (11) may give some information about the size of the Goldie constant which is of some value, especially in the situation when none of the existing formulae for it is satisfactory enough (see [7, 10, 6, 4]).

We can go even further. With a slight modification of the proof we can get rid of independence of X, B and obtain the following theorem.

Theorem 6. *Suppose that F is a separable Banach space. Let $p > 0$ and let an i.i.d. sequence $(X, B), (X_1, B_1), \ldots$ with values in $[0, \infty) \times F$ be such that X is nondegenerate and $\mathbb{E}\|B\|^p, \mathbb{E}X^p < \infty$. Assume additionally that*

$$\mathbb{P}(Xv + B = v) < 1 \text{ for every } v \in F. \tag{12}$$

Then there are constants $0 < c_p(X, B) \leq C_p(X, B) < \infty$ which depend on p and the distribution of (X, B) such that for every n,

$$c_p(X, B)\mathbb{E}\|B\|^p \sum_{i=1}^{n} \mathbb{E}R_{i-1}^p \leq \mathbb{E}\left\|\sum_{i=1}^{n} R_{i-1} B_i\right\|^p \leq C_p(X, B)\mathbb{E}\|B\|^p \sum_{i=1}^{n} \mathbb{E}R_{i-1}^p. \tag{13}$$

Theorem 6 specified to our situation with $\mathbb{E}X^p = 1$ says

$$c_p(X, B)\mathbb{E}\|B\|^p \leq \frac{1}{n} \mathbb{E}\|S_n\|^p \leq C_p(X, B)\mathbb{E}\|B\|^p, \tag{14}$$

This gives an estimate for the Goldie constant but now with $c_p(X, B), C_p(X, B)$ depending on the law of (X, B). Again, in particular cases, a careful examination of the constants involved in the proof may give a more satisfactory answer. Also, in view of Theorem 6, it would be worth relaxing the assumptions of [6].

The paper is organized as follows. In Section 2 and 3 we derive lower bounds in Theorems 2 and 3. Then in Section 4 we establish upper bounds in both theorems. We conclude in Section 5 with a discussion of the proof of Theorem 6.

2 Lower bound - $p > 1$

In this section we will show the lower bound in Theorem 3. Since it is only a matter of normalization we will assume that

$$X_1, X_2, \dots \text{ are independent, nonnegative r.v.'s such that } \mathbb{E}X_i^p = 1. \tag{15}$$

In particular this implies that $\mathbb{E}R_i^p = 1$ for all i.

We also set for $k = 1, 2, \dots$

$$R_{k,k-1} \equiv 1 \quad \text{and} \quad R_{k,i} := \prod_{j=k}^{i} X_i \text{ for } i \geq k.$$

Observe that $R_i = R_k R_{k+1,i}$ for $i \geq k \geq 0$.

We begin with several lemmas.

Lemma 7. *Suppose that a nonnegative random variable X satisfies $\mathbb{E}|X - \mathbb{E}X| \geq \mu$ and $\mathbb{E}|X - \mathbb{E}X|\mathbb{1}_{\{X>A\}} \leq \frac{1}{4}\mu$. Then for all $p \geq 1$ and $u, v \in (F, \| \ \|)$ we have*

$$\mathbb{E}\|uX + v\|^p \geq \mathbb{E}\|uX + v\|^p \mathbb{1}_{\{X \leq A\}} \geq \frac{\mu^p}{8^p} \min\left\{1, \frac{1}{(\mathbb{E}X)^p}\right\} \max\{\|u\|^p, \|v\|^p\}.$$

Proof. Let Y has the same distribution as X conditioned on the set $\{X \leq A\}$. Let us define $t := \mathbb{E}Y \leq \mathbb{E}X$. Clearly, $\mathbb{E}(X - \mathbb{E}X)_+ = \mathbb{E}(X - \mathbb{E}X)_- \geq \frac{1}{2}\mu$. Therefore,

$$\begin{aligned}
\mathbb{E}|X - t|\mathbb{1}_{\{X \leq A\}} &\geq \mathbb{E}(X - t)_+ \mathbb{1}_{\{X \leq A\}} \geq \mathbb{E}(X - \mathbb{E}X)_+ \mathbb{1}_{\{X \leq A\}} \\
&= \mathbb{E}(X - \mathbb{E}X)_+ - \mathbb{E}(X - \mathbb{E}X)_+ \mathbb{1}_{\{X > A\}} \\
&\geq \frac{1}{2}\mu - \mathbb{E}|X - \mathbb{E}X|\mathbb{1}_{\{X>A\}} \geq \frac{1}{2}\mu - \frac{1}{4}\mu = \frac{1}{4}\mu.
\end{aligned}$$

We obtain

$$\mathbb{E}\|uY + v\| = \frac{1}{t}\mathbb{E}\|v(t - Y) + (tu + v)Y\| \geq \frac{1}{t}\|v\|\mathbb{E}|Y - t| - \|tu + v\|.$$

Since $\mathbb{E}\|uY + v\| \geq \|u\mathbb{E}Y + v\| = \|tu + v\|$, we have

$$\mathbb{E}\|uY + v\| \geq \frac{1}{2t}\|v\|\mathbb{E}|Y - t| = \frac{\|v\|}{2t\mathbb{P}(X \leq A)}\mathbb{E}|X - t|\mathbb{1}_{\{X \leq A\}} \geq \frac{\mu}{8t}\frac{\|v\|}{\mathbb{P}(X \leq A)}.$$

We arrive at

$$\mathbb{E}\|uX+v\|^p\mathbb{1}_{\{X\le A\}} \ge \left(\mathbb{E}\|uX+v\|\mathbb{1}_{\{X\le A\}}\right)^p = (\mathbb{E}\|uY+v\|\mathbb{P}(X\le A))^p$$
$$\ge \frac{\mu^p}{8^p t^p}\|v\|^p \ge \frac{\mu^p}{8^p(\mathbb{E}X)^p}\|v\|^p.$$

We also have

$$\mathbb{E}\|uY+v\| = \mathbb{E}\|u(Y-t)+tu+v\| \ge \|u\|\mathbb{E}|Y-t| - \|tu+v\|.$$

Therefore

$$\mathbb{E}\|uY+v\| \ge \frac{\|u\|}{2}\mathbb{E}|Y-t| \ge \frac{\mu}{8}\frac{\|u\|}{\mathbb{P}(X\le A)}$$

and as before we get that $\mathbb{E}\|uX+v\|^p\mathbb{1}_{\{X\le A\}} \ge \frac{\mu^p}{8^p}\|u\|^p$. □

Lemma 8. *Assume that* (15) *and* (5) *hold. Then for any* $v_0, v_1, \ldots, v_n \in (F, \| \ \|)$ *we have*

$$\mathbb{E}\left\|\sum_{i=0}^n v_i R_i\right\|^p \ge \frac{\mu^{2p}}{64^p}\max_{1\le i\le n}\|v_i\|^p \ge \frac{\mu^{2p}}{64^p}\cdot\frac{1}{n}\sum_{i=1}^n\|v_i\|^p.$$

Proof. For $1 \le j \le n$ we have $\sum_{i=0}^n v_i R_i = Y + X_j(v_j R_{j-1} + X_{j+1}Z)$, where Y and Z are independent of X_j and X_{j+1}. Observe that $\mathbb{E}X_j \le 1$ and $\mathbb{E}X_{j+1} \le 1$. Thus, using Lemma 7 twice, we obtain

$$\mathbb{E}\left\|\sum_{i=0}^n v_i R_i\right\|^p \ge \frac{\mu^p}{8^p}\mathbb{E}\|v_j R_{j-1} + X_{j+1}Z\|^p \ge \frac{\mu^{2p}}{64^p}\|v_j\|^p\mathbb{E}|R_{j-1}|^p = \frac{\mu^{2p}}{64^p}\|v_j\|^p.$$

□

Lemma 9. *Assume that* (15) *holds and there exist* $q > 1$ *and* $0 < \lambda < 1$ *such that for all* i, $(\mathbb{E}X_i^q)^{1/q} \le \lambda$. *Then for any* $v_0, v_1, \ldots, v_n \in (F, \| \ \|)$ *and* $t > 0$,

$$\mathbb{P}\left(\left\|\sum_{i=0}^n v_i R_i\right\|^p \ge t\sum_{i=0}^n \lambda^i\|v_i\|^p\right) \le (1-\lambda)^{\frac{(1-p)q}{p}} t^{-\frac{q}{p}}.$$

Proof. Using Minkowski's and Hölder's inequalities we obtain

$$\left(\mathbb{E}\left\|\sum_{i=0}^n v_i R_i\right\|^q\right)^{\frac{1}{q}} \le \sum_{i=0}^n (\mathbb{E}\|v_i R_i\|^q)^{\frac{1}{q}} \le \sum_{i=0}^n \|v_i\|\lambda^i = \sum_{i=0}^n \|v_i\|\lambda^{\frac{i}{p}}\lambda^{\frac{p-1}{p}i}$$
$$\le \left(\sum_{k=0}^n \|v_i\|^p\lambda^i\right)^{\frac{1}{p}}\left(\sum_{i=0}^n \lambda^i\right)^{\frac{p-1}{p}}.$$

Thus,

$$\mathbb{E}\left\|\sum_{i=0}^{n} v_i R_i\right\|^q \leq \left(\sum_{i=0}^{n} \|v_i\|^p \lambda^i\right)^{\frac{q}{p}} (1-\lambda)^{-\frac{(p-1)q}{p}}.$$

By Chebyshev's inequality we get

$$\mathbb{P}\left(\left\|\sum_{i=0}^{n} v_i R_i\right\|^q \geq t^{\frac{q}{p}} \left(\sum_{i=0}^{n} \lambda^i \|v_i\|^p\right)^{\frac{q}{p}}\right) \leq (1-\lambda)^{\frac{(1-p)q}{p}} t^{-\frac{q}{p}}.$$

□

Lemma 10. *Let Y, Z be random vectors with values in a normed space F and let $p \geq 1$. Suppose that $\mathbb{E}\|Y\|^{p-1}\|Z\| \leq \gamma \mathbb{E}\|Z\|^p$. Then*

$$\mathbb{E}\|Y+Z\|^p \geq \mathbb{E}\|Y\|^p + \left(\frac{1}{3^p} - 2p\gamma\right) \mathbb{E}\|Z\|^p.$$

Proof. For any real numbers a, b we have $|a+b|^p \geq |a|^p - p|a|^{p-1}|b|$. If, additionally, $|a| \leq \frac{1}{3}|b|$, then $|a+b|^p \geq |a|^p + \frac{1}{3^p}|b|^p$. Taking $a = \|Y\|$, $b = -\|Z\|$ and using the inequality $\|Y+Z\| \geq |\|Y\| - \|Z\||$ we obtain

$$\begin{aligned}
\mathbb{E}\|Y+Z\|^p &= \mathbb{E}\|Y+Z\|^p \mathbb{1}_{\{\|Y\| \leq \frac{1}{3}\|Z\|\}} + \mathbb{E}\|Y+Z\|^p \mathbb{1}_{\{\|Y\| > \frac{1}{3}\|Z\|\}} \\
&\geq \mathbb{E}\|Y\|^p \mathbb{1}_{\{\|Y\| \leq \frac{1}{3}\|Z\|\}} + \frac{1}{3^p} \mathbb{E}\|Z\|^p \mathbb{1}_{\{\|Y\| \leq \frac{1}{3}\|Z\|\}} \\
&\quad + \mathbb{E}\|Y\|^p \mathbb{1}_{\{\|Y\| > \frac{1}{3}\|Z\|\}} - p\mathbb{E}\|Y\|^{p-1}\|Z\| \mathbb{1}_{\{\|Y\| > \frac{1}{3}\|Z\|\}} \\
&= \mathbb{E}\|Y\|^p + \frac{1}{3^p} \mathbb{E}\|Z\|^p (1 - \mathbb{1}_{\{\|Y\| > \frac{1}{3}\|Z\|\}}) - p\mathbb{E}\|Y\|^{p-1}\|Z\| \mathbb{1}_{\{\|Y\| > \frac{1}{3}\|Z\|\}}.
\end{aligned}$$

Note that

$$\mathbb{E}\left(\frac{1}{3^p}\|Z\|^p + p\mathbb{E}\|Y\|^{p-1}\|Z\|\right) \mathbb{1}_{\{\|Y\| > \frac{1}{3}\|Z\|\}} \leq \left(\frac{1}{3} + p\right) \mathbb{E}\|Y\|^{p-1}\|Z\| \leq 2p\gamma \mathbb{E}\|Z\|^p.$$

Therefore,

$$\mathbb{E}\|Y+Z\|^p \geq \mathbb{E}\|Y\|^p + \frac{1}{3^p}\mathbb{E}\|Z\|^p - 2p\gamma \mathbb{E}\|Z\|^p.$$

□

We are now able to state the key proposition which will easily yield the lower bound in Theorem 3.

Proposition 11. *Let $p>1$ and r.v.'s $X_1, X_2, \ldots$ satisfy assumptions* (15), (5) *and* (6). *Then there exist constants $\varepsilon_0, \varepsilon_1, C_0 > 0$ depending only on p, μ, A, q and λ such that for any vectors $v_0, v_1, \ldots, v_n$ in a normed space $(F, \| \ \|)$ and $k \geq 1$ we have*

$$\mathbb{E}\left\|\sum_{i=0}^{n} v_i R_i\right\|^p \geq \varepsilon_0 \|v_0\|^p + \sum_{i=1}^{n}\left(\frac{\varepsilon_1}{k} - c_i\right)\|v_i\|^p, \tag{16}$$

where

$$c_i = 0 \ \text{ for } 1 \leq i \leq k-1, \quad c_i = \Phi \sum_{j=k}^{i} \lambda^j \ \text{ for } i \geq k \quad \text{and} \quad \Phi = C_0 \lambda^{(p-1)k}.$$

Proof. Define

$$\varepsilon_0 := \min\left\{\frac{1}{4\cdot 3^p}, \frac{\mu^p}{8\cdot 24^p}\right\}, \quad \varepsilon_1 := \min\left\{\frac{\mu^p}{8^p}, \frac{\mu^{2p}}{2^{p-1}64^p}\right\}\varepsilon_0,$$

the value of C_0 will be chosen later. In the proof by ε_2, C_2, C_3 we will denote finite nonnegative constants that depend only on parameters $p, \mu.A, q$ and λ.

We fix $k \geq 1$ and prove (16) by induction on n. From Lemma 7 and Lemma 8 we obtain

$$\mathbb{E}\left\|\sum_{i=0}^{n} v_i R_i\right\|^p \geq 2\varepsilon_0 \|v_0\|^p, \qquad \mathbb{E}\left\|\sum_{i=0}^{n} v_i R_i\right\|^p \geq \frac{2\varepsilon_1}{n}\sum_{i=1}^{n}\|v_i\|^p.$$

Therefore for $n \leq k$ we have

$$\mathbb{E}\left\|\sum_{i=0}^{n} v_i R_i\right\|^p \geq \varepsilon_0 \|v_0\|^p + \frac{\varepsilon_1}{k}\sum_{i=1}^{n}\|v_i\|^p.$$

Suppose that the induction assertion holds for $n \geq k$. We show it for $n+1$. To this end we consider two cases.

Case 1. $\varepsilon_0 \|v_0\|^p \leq \Phi \sum_{i=k}^{n+1} \lambda^i \|v_i\|^p$.

Applying the induction assumption conditionally on X_1 we obtain

$$\begin{aligned}
\mathbb{E}\left\|\sum_{i=0}^{n+1} v_i R_i\right\|^p &\geq \varepsilon_0 \mathbb{E}\|v_0 + v_1 X_1\|^p + \sum_{i=2}^{n+1}\left(\frac{\varepsilon_1}{k} - c_{i-1}\right)\mathbb{E}\|X_1 v_i\|^p \\
&\geq \frac{\varepsilon_1}{k}\|v_1\|^p + \sum_{i=2}^{n+1}\left(\frac{\varepsilon_1}{k} - c_{i-1}\right)\|v_i\|^p \\
&\geq \varepsilon_0\|v_0\|^p - \Phi\sum_{i=k}^{n+1}\lambda^i\|v_i\|^p + \frac{\varepsilon_1}{k}\|v_1\|^p + \sum_{i=2}^{n+1}\left(\frac{\varepsilon_1}{k} - c_{i-1}\right)\|v_i\|^p \\
&= \varepsilon_0\|v_0\|^p + \sum_{i=1}^{n+1}\left(\frac{\varepsilon_1}{k} - c_i\right)\|v_i\|^p,
\end{aligned}$$

where the second inequality follows from Lemma 7.

Case 2. $\varepsilon_0\|v_0\|^p > \Phi\sum_{i=k}^{n+1}\lambda^i\|v_i\|^p$.

Define the event $A_k \in \sigma(X_1,\ldots,X_k)$ by

$$A_k := \{X_1 \le A, R_{2,k} \le 2^{\frac{1}{q}}\lambda^{k-1}\}.$$

By the induction assumption used conditionally on $X_1,\ldots,X_k$ we have

$$\mathbb{E}\left\|\sum_{i=0}^{n+1} v_iR_i\right\|^p \mathbb{1}_{\Omega\setminus A_k} \ge \varepsilon_0\mathbb{E}\left\|\sum_{i=0}^{k} v_iR_i\right\|^p \mathbb{1}_{\Omega\setminus A_k} + \sum_{i=k+1}^{n+1}\left(\frac{\varepsilon_1}{k} - c_{i-k}\right)\mathbb{E}\|v_iR_k\|^p\mathbb{1}_{\Omega\setminus A_k}. \quad (17)$$

We have by Chebyshev's inequality and (6),

$$\mathbb{P}\left(R_{2,k} \ge 2^{\frac{1}{q}}\lambda^{k-1}\right) \le \frac{\mathbb{E}R_{2,k}^q}{2\lambda^{(k-1)q}} \le \frac{1}{2}. \quad (18)$$

Together with (5) it implies $\mathbb{P}(A_k) > 0$. Let (Y, Y', Z) have the same distribution as the random vector $(\sum_{i=k}^{n+1} v_iR_i, \sum_{i=k}^{n+1} v_iR_{k+1,i}, \sum_{i=0}^{k-1} v_iR_i)$ conditioned on the event A_k. Note that

$$\mathbb{E}\left\|\sum_{i=0}^{n+1} v_iR_i\right\|^p \mathbb{1}_{A_k} = \mathbb{P}(A_k)\mathbb{E}\|Y+Z\|^p.$$

Applying Lemma 7 conditionally we obtain

$$\begin{aligned}\mathbb{E}\|Z\|^p &= \frac{1}{\mathbb{P}(A_k)}\mathbb{E}\left\|\sum_{i=0}^{k-1} v_iR_i\right\|^p \mathbb{1}_{\{X_1\le A\}}\mathbb{1}_{\{R_{2,k}\le 2^{\frac{1}{q}}\lambda^{k-1}\}}\\ &\ge \frac{\mu^p}{8^p}\|v_0\|^p\frac{\mathbb{P}(R_{2,k}\le 2^{\frac{1}{q}}\lambda^{k-1})}{\mathbb{P}(A_k)} = \frac{\mu^p}{8^p}\|v_0\|^p\frac{1}{\mathbb{P}(X_1\le A)} \ge \frac{\mu^p}{8^p}\|v_0\|^p. \quad (19)\end{aligned}$$

Note that Y' has the same distribution as $\sum_{i=k}^{n+1} v_iR_{k+1,i}$ and is independent of Z. We have for $t > 0$,

$$\begin{aligned}\mathbb{P}(\|Y\|^p \ge t\mathbb{E}\|Z\|^p) &\le \mathbb{P}\left(A^p\lambda^{p(k-1)}2^{\frac{p}{q}}\|Y'\|^p \ge t\frac{\mu^p}{8^p}\|v_0\|^p\right)\\ &\le \mathbb{P}\left(A^p\lambda^{p(k-1)}2^{\frac{p}{q}}\|Y'\|^p \ge t\frac{\mu^p}{8^p}\frac{\Phi}{\varepsilon_0}\sum_{i=k}^{n+1}\lambda^i\|v_i\|^p\right)\\ &= \mathbb{P}\left(\|Y'\|^p \ge tC_0\varepsilon_2\sum_{i=k}^{n+1}\lambda^{i-k}\|v_i\|^p\right) \le C_1(tC_0)^{-\frac{q}{p}}, \quad (20)\end{aligned}$$

where the last inequality follows by Lemma 9 (recall that ε_2 and C_1 denote constants depending on p, μ, A, q and λ).

In order to use Lemma 10 we would like to estimate $\mathbb{E}\|Y\|^{p-1}\|Z\|$. To this end take $\delta > 0$ and observe first that

$$\mathbb{E}\|Y\|^{p-1}\|Z\| \leq \mathbb{E}\|Y\|^{p-1}\|Z\|\mathbb{1}_{\{\|Y\|^p \leq \delta\mathbb{E}\|Z\|^p\}} + \mathbb{E}\|Y\|^{p-1}\|Z\|\mathbb{1}_{\{\|Z\|^p \leq \delta\mathbb{E}\|Z\|^p\}} + \mathbb{E}\|Y\|^{p-1}\|Z\|\mathbb{1}_{\{\|Y\|^p > \delta\mathbb{E}\|Z\|^p\}}\mathbb{1}_{\{\|Z\|^p > \delta\mathbb{E}\|Z\|^p\}}. \tag{21}$$

Clearly,

$$\mathbb{E}\|Y\|^{p-1}\|Z\|\mathbb{1}_{\{\|Y\|^p \leq \delta\mathbb{E}\|Z\|^p\}} \leq \delta^{\frac{p-1}{p}} (\mathbb{E}\|Z\|^p)^{\frac{p-1}{p}} \mathbb{E}\|Z\| \leq \delta^{\frac{p-1}{p}} \mathbb{E}\|Z\|^p. \tag{22}$$

To estimate the next term in (21) note that

$$\mathbb{E}\|Y\|^{p-1}\|Z\|\mathbb{1}_{\{\|Z\|^p \leq \delta\mathbb{E}\|Z\|^p\}} \leq \delta^{1/p}(\mathbb{E}\|Z\|^p)^{1/p}\mathbb{E}\|Y\|^{p-1}.$$

Using estimate (20) we obtain

$$\begin{aligned}\mathbb{E}\|Y\|^{p-1} &= (\mathbb{E}\|Z\|^p)^{\frac{p-1}{p}} \int_0^\infty \mathbb{P}\left(\|Y\|^p \geq s^{\frac{p}{p-1}}\mathbb{E}\|Z\|^p\right) \mathrm{d}s \\ &\leq (\mathbb{E}\|Z\|^p)^{\frac{p-1}{p}} \int_0^\infty \min\{1, C_1 C_0^{-\frac{q}{p}} s^{-\frac{q}{p-1}}\} \,\mathrm{d}s \leq (\mathbb{E}\|Z\|^p)^{\frac{p-1}{p}} \left(1 + C_2 C_0^{-\frac{q}{p}}\right),\end{aligned}$$

where the last inequality follows since $q > p-1$. Thus,

$$\mathbb{E}\|Y\|^{p-1}\|Z\|\mathbb{1}_{\{\|Z\|^p \leq \delta\mathbb{E}\|Z\|^p\}} \leq \delta^{1/p}\left(1 + C_2 C_0^{-\frac{q}{p}}\right)\mathbb{E}\|Z\|^p. \tag{23}$$

We are left with estimating the last term in (21). We have

$$\begin{aligned}&\mathbb{E}\|Y\|^{p-1}\|Z\|\mathbb{1}_{\{\|Y\|^p > \delta\mathbb{E}\|Z\|^p\}}\mathbb{1}_{\{\|Z\|^p > \delta\mathbb{E}\|Z\|^p\}} \\ &\quad = \sum_{m=0}^\infty \mathbb{E}\|Y\|^{p-1}\|Z\|\mathbb{1}_{\{2^m\delta\mathbb{E}\|Z\|^p < \|Y\|^p \leq 2^{m+1}\delta\mathbb{E}\|Z\|^p\}}\mathbb{1}_{\{\|Z\|^p > \delta\mathbb{E}\|Z\|^p\}} \\ &\quad \leq \sum_{m=0}^\infty 2^{(m+1)\frac{p-1}{p}}\delta^{\frac{p-1}{p}}\mathbb{E}(\mathbb{E}\|Z\|^p)^{\frac{p-1}{p}}\|Z\|\mathbb{1}_{\{2^m\delta\mathbb{E}\|Z\|^p < \|Y\|^p\}}\mathbb{1}_{\{\|Z\|^p > \delta\mathbb{E}\|Z\|^p\}} \\ &\quad \leq \delta^{\frac{p-1}{p}} \sum_{m=0}^\infty 2^{(m+1)\frac{p-1}{p}}\mathbb{E}\left(\frac{\|Z\|^p}{\delta}\right)^{\frac{p-1}{p}}\|Z\|\mathbb{1}_{\{2^m\delta\mathbb{E}\|Z\|^p < \|Y\|^p\}} \\ &\quad = \sum_{m=0}^\infty 2^{(m+1)\frac{p-1}{p}}\mathbb{E}\|Z\|^p\mathbb{1}_{\{2^m\delta\mathbb{E}\|Z\|^p < \|Y\|^p\}}.\end{aligned}$$

Recall that Z and Y' are independent. Therefore as in (20) we get

$$\begin{aligned}\mathbb{E}\|Z\|^p\mathbb{1}_{\{2^m\delta\mathbb{E}\|Z\|^p < \|Y\|^p\}} &\leq \mathbb{E}\|Z\|^p\mathbb{1}_{\{\|Y'\|^p \geq 2^m\delta C_0\varepsilon_2\sum_{i=k}^{n+1}\lambda^{i-k}\|v_i\|^p\}} \\ &\leq \mathbb{E}\|Z\|^p\mathbb{P}\left(\|Y'\|^p \geq 2^m\delta C_0\varepsilon_2\sum_{i=k}^{n+1}\lambda^{i-k}\|v_i\|^p\right) \\ &\leq \mathbb{E}\|Z\|^p C_1(2^m\delta C_0)^{-\frac{q}{p}}.\end{aligned}$$

We arrive at

$$\mathbb{E}\|Y\|^{p-1}\|Z\|\mathbb{1}_{\{\|Y\|^p>\delta\mathbb{E}\|Z\|^p\}}\mathbb{1}_{\{\|Z\|^p>\delta\mathbb{E}\|Z\|^p\}} \le \mathbb{E}\|Z\|^p C_1(\delta C_0)^{-\frac{q}{p}}\sum_{m=0}^{\infty} 2^{(m+1)\frac{p-1}{p}}2^{-\frac{mq}{p}}$$
$$\le \mathbb{E}\|Z\|^p C_3(\delta C_0)^{-\frac{q}{p}}, \tag{24}$$

where we have used the fact that $q > p-1$.

Estimates (21)–(24) imply

$$\mathbb{E}\|Y\|^{p-1}\|Z\| \le \mathbb{E}\|Z\|^p\left(\delta^{\frac{p-1}{p}} + \delta^{1/p}(1+C_2C_0^{-\frac{q}{p}}) + C_3(\delta C_0)^{-\frac{q}{p}}\right).$$

Now we choose $\delta = \delta(p)$ sufficiently small and then $C_0 = C_0(p, A, \mu.q, \lambda)$ sufficiently large to obtain

$$\mathbb{E}\|Y\|^{p-1}\|Z\| \le \frac{1}{4p3^p}\mathbb{E}\|Z\|^p. \tag{25}$$

From Lemma 10 we deduce

$$\mathbb{E}\|Y+Z\|^p \ge \mathbb{E}\|Y\|^p + \frac{1}{2\cdot 3^p}\mathbb{E}\|Z\|^p.$$

Hence

$$\mathbb{E}\left\|\sum_{i=0}^{n+1} v_iR_i\right\|^p \mathbb{1}_{A_k} \ge \frac{1}{2\cdot 3^p}\mathbb{E}\left\|\sum_{i=0}^{k-1} v_iR_i\right\|^p \mathbb{1}_{A_k} + \mathbb{E}\left\|\sum_{i=k}^{n+1} v_iR_i\right\|^p \mathbb{1}_{A_k}. \tag{26}$$

Lemma 7 and (18) yield

$$\mathbb{E}\left\|\sum_{i=0}^{k-1} v_iR_i\right\|^p \mathbb{1}_{A_k} \ge \frac{\mu^p}{8^p}\|v_0\|^p\mathbb{P}(R_{2,k} \le 2^{\frac{1}{q}}\lambda^{k-1}) \ge \frac{1}{2}\cdot\frac{\mu^p}{8^p}\|v_0\|^p.$$

It follows that

$$\frac{1}{2\cdot 3^p}\mathbb{E}\left\|\sum_{i=0}^{k-1} v_iR_i\right\|^p \mathbb{1}_{A_k} \ge \varepsilon_0\|v_0\|^p + \varepsilon_0\mathbb{E}\left\|\sum_{i=0}^{k-1} v_iR_i\right\|^p \mathbb{1}_{A_k}. \tag{27}$$

By the induction assumption we obtain

$$\mathbb{E}\left\|\sum_{i=k}^{n+1} v_iR_i\right\|^p \mathbb{1}_{A_k} \ge \varepsilon_0\mathbb{E}\|v_kR_k\|^p\mathbb{1}_{A_k} + \sum_{i=k+1}^{n+1}\left(\frac{\varepsilon_1}{k} - c_{i-k}\right)\mathbb{E}\|v_iR_k\|^p\mathbb{1}_{A_k}. \tag{28}$$

Combining (26), (27) and (28) we get

$$\begin{aligned}
\mathbb{E}\left\|\sum_{i=0}^{n+1} v_i R_i\right\|^p \mathbb{1}_{A_k} &\geq \varepsilon_0 \|v_0\|^p + \varepsilon_0 \mathbb{E}\left\|\sum_{i=0}^{k-1} v_i R_i\right\|^p \mathbb{1}_{A_k} + \varepsilon_0 \mathbb{E}\|v_k R_k\|^p \mathbb{1}_{A_k} \\
&\quad + \sum_{i=k+1}^{n+1} \left(\frac{\varepsilon_1}{k} - c_{i-k}\right) \mathbb{E}\|v_i R_k\|^p \mathbb{1}_{A_k} \\
&\geq \varepsilon_0 \|v_0\|^p + \frac{\varepsilon_0}{2^{p-1}} \mathbb{E}\left\|\sum_{i=0}^{k} v_i R_i\right\|^p \mathbb{1}_{A_k} + \sum_{i=k+1}^{n+1} \left(\frac{\varepsilon_1}{k} - c_{i-k}\right) \mathbb{E}\|v_i R_k\|^p \mathbb{1}_{A_k}.
\end{aligned}$$

This inequality together with (17) and Lemma 8 yields

$$\begin{aligned}
\mathbb{E}\left\|\sum_{i=0}^{n+1} v_i R_i\right\|^p &\geq \varepsilon_0 \|v_0\|^p + \frac{\varepsilon_0}{2^{p-1}} \mathbb{E}\left\|\sum_{i=0}^{k} v_i R_i\right\|^p + \sum_{i=k+1}^{n+1} \left(\frac{\varepsilon_1}{k} - c_{i-k}\right) \mathbb{E}\|v_i R_k\|^p \\
&\geq \varepsilon_0 \|v_0\|^p + \frac{\varepsilon_1}{k} \sum_{i=1}^{k} \|v_i\|^p + \sum_{i=k+1}^{n+1} \left(\frac{\varepsilon_1}{k} - c_{i-k}\right) \|v_i\|^p \\
&\geq \varepsilon_0 \|v_0\|^p + \sum_{i=1}^{n+1} \left(\frac{\varepsilon_1}{k} - c_i\right) \|v_i\|^p.
\end{aligned}$$

□

We are ready to prove the lower L_p-estimate for $p > 1$.

Proof of the lower bound in Theorem 3. For sufficiently large k we have for all i,

$$c_i \leq \frac{\Phi \lambda^k}{1-\lambda} = \frac{C_0 \lambda^{pk}}{1-\lambda} \leq \frac{\varepsilon_1}{2k}.$$

Thus, Proposition 11 yields

$$\mathbb{E}\left\|\sum_{i=0}^{n} v_i R_i\right\|^p \geq \varepsilon_0 \|v_0\|^p + \frac{\varepsilon_1}{2k} \sum_{i=1}^{n} \|v_i\|^p \geq \varepsilon \sum_{i=0}^{n} \|v_i\|^p,$$

where $\varepsilon := \min\{\varepsilon_0, \frac{\varepsilon_1}{2k}\}$. □

Remark. Observe that $\mu \leq \mathbb{E}|X_i - \mathbb{E}X_i| \leq 2\mathbb{E}X_i \leq 2(\mathbb{E}X_i^p)^{1/p} = 2$. This shows that

$$\varepsilon_0 = \frac{\mu^p}{8 \cdot 24^p}, \quad \varepsilon_1 = \frac{\mu^{2p}}{2^{p-1} 64^p} \cdot \varepsilon_0 \quad \text{and} \quad \min\left\{\varepsilon_0, \frac{\varepsilon_1}{2k}\right\} = \frac{\mu^{3p}}{8k \cdot 2^{10p} \cdot 3^p}.$$

Other constants used in the proof of Proposition 11 may be estimated as follows

$$\varepsilon_2 = \frac{\mu^p \lambda^p}{8^p A^p \varepsilon_0} 2^{-\frac{p}{q}} \geq \left(\frac{3\lambda}{2A}\right)^p, \quad C_1 = (1-\lambda)^{\frac{(1-p)q}{p}} \varepsilon_2^{-\frac{q}{p}} \leq (1-\lambda)^{\frac{(1-p)q}{p}} \left(\frac{2A}{3\lambda}\right)^q,$$

$$C_2 \leq \frac{p-1}{q+1-p} C_1 \quad \text{and} \quad C_3 = \frac{2^{\frac{q}{p}}}{2^{\frac{q+1-p}{p}} - 1} C_1 \leq \frac{2p}{(q+1-p)\ln 2} C_1.$$

Hence we can for example take

$$\delta := 48^{-\frac{p^2}{\min\{p-1,1\}}} \quad \text{and} \quad C_0 := (1-\lambda)^{1-p} \left(\frac{2A}{3\lambda}\right)^p \left(\frac{2p}{(q+1-p)\ln 2}\right)^{\frac{p}{q}} 48^{\frac{2p^2}{\min\{p-1,1\}}},$$

then each term $\delta^{(p-1)/p}$, $\delta^{1/p}$, $\delta^{1/p} C_2 C_0^{-q/p}$ and $C_3(\delta C_0)^{-q/p}$ is not greater than $48^{-p} \leq (16p3^p)^{-1}$ and (25) holds.

3 Lower bound - $p \leq 1$

In this section we prove the lower bound in Theorem 2. We will also assume normalization (15) and use similar notation as for $p > 1$.

We begin with a result similar to Lemma 7.

Lemma 12. *Let X be a nonnegative random variable such that $\mathbb{E}X^p = 1$. Then for every $A > 1$ and u, v in a normed space $(F, \| \ \|)$ we have*

$$\mathbb{E}\|uX + v\|^p \geq \mathbb{E}\|uX + v\|^p \mathbb{1}_{\{X^p \leq A\}} \geq \delta \max\{\|u\|^p, \|v\|^p\},$$

where

$$\delta := \mathbb{E}(X^p - 1)\mathbb{1}_{\{1 \leq X^p \leq A\}}.$$

Proof. Since $\mathbb{E}X^p = 1$ we have

$$\begin{aligned}\delta &= \mathbb{E}(X^p - 1)\mathbb{1}_{\{1 \leq X^p \leq A\}} \leq \mathbb{E}(X^p - 1)\mathbb{1}_{\{1 \leq X^p\}} \\ &= \mathbb{E}(1 - X^p)\mathbb{1}_{\{X^p \leq 1\}} \leq \mathbb{P}(X^p \leq 1) \leq \mathbb{P}(X^p \leq A). \end{aligned} \tag{29}$$

The triangle inequality yields $\|uX + v\| \geq \big|\|u\|X - \|v\|\big|$. Thus, it suffices to prove

$$\mathbb{E}\big|\|u\|X - \|v\|\big|^p \mathbb{1}_{\{X^p \leq A\}} \geq \delta \max\{\|u\|^p, \|v\|^p\}. \tag{30}$$

If $u = 0$ then this inequality is satisfied due to (29). In the case $u \neq 0$ divide both sides of (30) by $\|u\|^p$ to see that it is enough to show

$$\mathbb{E}|X - t|^p \mathbb{1}_{\{X^p \leq A\}} \geq \delta \max\{t^p, 1\} \quad \text{for } t \geq 0.$$

To prove this inequality let us consider two cases. First assume that $t \in [0,1]$. Then we have

$$\begin{aligned}\mathbb{E}|X-t|^p \mathbb{1}_{\{X^p \le A\}} &\ge \mathbb{E}|X-t|^p \mathbb{1}_{\{1\le X^p \le A\}} \ge \mathbb{E}(X^p - t^p)\mathbb{1}_{\{1 \le X^p \le A\}} \\ &\ge \mathbb{E}(X^p - 1)\mathbb{1}_{\{1\le X^p\le A\}} = \delta = \delta \max\{t^p, 1\}.\end{aligned}$$

In the case $t > 1$ it suffices to note that

$$\begin{aligned}\mathbb{E}|X-t|^p \mathbb{1}_{\{X^p \le A\}} &\ge \mathbb{E}|X-t|^p \mathbb{1}_{\{X^p \le 1\}} \ge \mathbb{E}(t^p - X^p)\mathbb{1}_{\{X^p \le 1\}} \\ &\ge \mathbb{E}(t^p - t^p X^p)\mathbb{1}_{\{X^p \le 1\}} = t^p \mathbb{E}(1-X^p)\mathbb{1}_{\{X^p\le 1\}} \ge \delta t^p = \delta \max\{t^p, 1\},\end{aligned}$$

where the last inequality follows from (29). □

As a consequence, in the same way as in Lemma 8, we derive the following estimate.

Lemma 13. *Let r.v.'s $X_1, X_2, \ldots$ satisfy* (15) *and* (4)*. Then for any vectors $v_0, v_1, \ldots, v_n \in F$ we get*

$$\mathbb{E}\left\|\sum_{i=0}^n v_i R_i\right\|^p \ge \delta^2 \max_{1\le i\le n} \|v_i\|^p \ge \frac{\delta^2}{n}\sum_{i=1}^n \|v_i\|^p.$$

Lemma 14. *Suppose that random variables $X_1, X_2, \ldots$ satisfy assumptions* (15) *and* (3)*. Then for all vectors $v_1, v_2, \ldots$ in $(F, \|\ \|)$ we have*

$$\mathbb{P}\left(\left\|\sum_{i=0}^n v_i R_i\right\|^p \ge \frac{t}{1-\lambda}\sum_{i=0}^n \lambda^i \|v_i\|^p\right) \le \frac{1}{\sqrt{t}} \quad \textit{for } t > 0.$$

Proof. Note that

$$\mathbb{E}\left\|\sum_{i=0}^n v_i R_i\right\|^{p/2} \le \sum_{i=0}^n \|v_i\|^{p/2}\mathbb{E}R_i^{p/2} \le \sum_{i=0}^n \lambda^i \|v_i\|^{p/2}.$$

By the Cauchy-Schwarz inequality we get

$$\left(\sum_{i=0}^n \lambda^i \|v_i\|^{p/2}\right)^2 \le \sum_{i=0}^n \lambda^i \sum_{i=0}^n \lambda^i \|v_i\|^p \le \frac{1}{1-\lambda}\sum_{i=0}^n \lambda^i \|v_i\|^p.$$

Thus, using Chebyshev's inequality we arrive at

$$\begin{aligned}\mathbb{P}\left(\left\|\sum_{i=0}^n v_i R_i\right\|^p \ge \frac{t}{1-\lambda}\sum_{i=0}^n \lambda^i\|v_i\|^p\right) &\le \mathbb{P}\left(\left\|\sum_{i=0}^n v_i R_i\right\|^{p/2} \ge \sqrt{t}\sum_{i=0}^n \lambda^i \|v_i\|^{p/2}\right) \\ &\le \left(\sqrt{t}\sum_{i=0}^n \lambda^i \|v_i\|^{p/2}\right)^{-1} \mathbb{E}\left\|\sum_{i=0}^n v_i R_i\right\|^{p/2} \le \frac{1}{\sqrt{t}}.\end{aligned}$$

□

Our next lemma is in the spirit of Lemma 10, but it has a simpler proof.

Lemma 15. *Let Y, Z be random vectors with values in a normed space $(F, \| \ \|)$ such that*

$$\mathbb{E}\|Z\|^p \mathbb{1}_{\{\|Y\|^p \geq \frac{1}{8}\mathbb{E}\|Z\|^p\}} \leq \frac{1}{8}\mathbb{E}\|Z\|^p.$$

Then

$$\mathbb{E}\|Y+Z\|^p \geq \mathbb{E}\|Y\|^p + \frac{1}{2}\mathbb{E}\|Z\|^p.$$

Proof. We have for any $u, v \in F$, $\|u+v\|^p \geq \big|\|u\| - \|v\|\big|^p \geq \|u\|^p - \|v\|^p$, therefore

$$\begin{aligned}\mathbb{E}\|Y+Z\|^p &\geq \mathbb{E}(\|Y\|^p + \|Z\|^p - 2\|Z\|^p)\mathbb{1}_{\{\|Y\|^p \geq \frac{1}{8}\mathbb{E}\|Z\|^p\}} \\ &\quad + \mathbb{E}(\|Y\|^p + \|Z\|^p - 2\|Y\|^p)\mathbb{1}_{\{\|Y\|^p < \frac{1}{8}\mathbb{E}\|Z\|^p\}} \\ &\geq \mathbb{E}\|Y\|^p + \mathbb{E}\|Z\|^p - 2\mathbb{E}\|Z\|^p\mathbb{1}_{\{\|Y\|^p \geq \frac{1}{8}\mathbb{E}\|Z\|^p\}} - 2\mathbb{E}\|Y\|^p\mathbb{1}_{\{\|Y\|^p < \frac{1}{8}\mathbb{E}\|Z\|^p\}} \\ &\geq \mathbb{E}\|Y\|^p + \mathbb{E}\|Z\|^p - 2\cdot\frac{1}{8}\mathbb{E}\|Z\|^p - 2\cdot\frac{1}{8}\mathbb{E}\|Z\|^p = \mathbb{E}\|Y\|^p + \frac{1}{2}\mathbb{E}\|Z\|^p.\end{aligned}$$

□

The proof of the lower bound for $p \leq 1$ is similar to the proof for $p > 1$ and it relies on a proposition similar to Proposition 11.

Proposition 16. *Let $0 < p \leq 1$ and r.v.'s $X_1, X_2, \ldots$ satisfy assumptions* (15), (3) *and* (4). *Then for any vectors $v_0, v_1, \ldots, v_n$ in a normed space $(F, \| \ \|)$ and any integer $k \geq 1$ we have*

$$\mathbb{E}\left\|\sum_{i=0}^{n} v_i R_i\right\|^p \geq \varepsilon_0\|v_0\|^p + \sum_{i=1}^{n}\left(\frac{\varepsilon_1}{k} - c_i\right)\|v_i\|^p,$$

where $\varepsilon_0 = \delta/8$, $\varepsilon_1 = \delta^3/8$ and

$$c_i = 0 \text{ for } 1 \leq i \leq k-1, \quad c_i = \Phi\sum_{j=k}^{i}\lambda^j \text{ for } i \geq k \quad \text{and} \quad \Phi = \frac{2^8 A}{1-\lambda}\lambda^{k-2}.$$

Proof. For $n \leq k$ the assertion follows by Lemmas 12 and 13, since $\varepsilon_0 \leq \delta/2$ and $\varepsilon_1/k \leq \varepsilon_1/n \leq \delta^2/(2n)$. For $n \geq k$ we proceed by induction on n.

Case 1. $\varepsilon_0\|v_0\|^p \leq \Phi\sum_{i=k}^{n+1}\lambda^i\|v_i\|^p$.
In this case the induction step is the same as in the proof of Proposition 11.

Case 2. $\varepsilon_0\|v_0\|^p > \Phi\sum_{i=k}^{n+1}\lambda^i\|v_i\|^p$.
Let us define the set

$$A_k := \{X_1^p \leq A, R_{2,k}^p \leq 4\lambda^{2k-2}\}.$$

By the induction hypothesis we have

$$\mathbb{E}\left\|\sum_{i=0}^{n+1} v_iR_i\right\|^p \mathbb{1}_{\Omega\setminus A_k} \geq \varepsilon_0\mathbb{E}\left\|\sum_{i=0}^{k} v_iR_i\right\|^p \mathbb{1}_{\Omega\setminus A_k} + \sum_{i=k+1}^{n+1}\left(\frac{\varepsilon_1}{k} - c_{i-k}\right)\mathbb{E}\|v_iR_k\|^p\mathbb{1}_{\Omega\setminus A_k}. \quad (31)$$

By Chebyshev's inequality and (3) we get

$$\mathbb{P}(R_{2,k}^p > 4\lambda^{2k-2}) \leq \frac{\mathbb{E}R_{2,k}^{p/2}}{2\lambda^{k-1}} \leq \frac{1}{2}, \quad (32)$$

in particular $\mathbb{P}(A_k) > 0$. Let Y, Y', Z be defined as in the proof of Proposition 11. As in (19) we show that Lemma 12 yields $\mathbb{E}\|Z\|^p \geq \delta\|v_0\|^p$. We have $\|Y\|^p \leq 4A\lambda^{2k-2}\|Y'\|^p$, variables Y' and Z are independent and Y' has the same distribution as $\sum_{i=k}^{n+1} v_iR_{k+1,i}$. Thus,

$$\begin{aligned}
\mathbb{E}\|Z\|^p\mathbb{1}_{\{\|Y\|^p\geq\frac{1}{8}\mathbb{E}\|Z\|^p\}} &\leq \mathbb{E}\|Z\|^p\mathbb{1}_{\{4A\lambda^{2k-2}\|Y'\|^p\geq\frac{\delta}{8}\|v_0\|^p\}} \\
&= \mathbb{E}\|Z\|^p\mathbb{P}\left(\|Y'\|^p \geq \frac{1}{4A\lambda^{2k-2}}\varepsilon_0\|v_0\|^p\right) \\
&\leq \mathbb{E}\|Z\|^p\mathbb{P}\left(\left\|\sum_{i=k}^{n+1} v_iR_{k+1,i}\right\|^p \geq \frac{2^6}{1-\lambda}\sum_{i=k}^{n+1}\lambda^{i-k}\|v_i\|^p\right) \\
&\leq \frac{1}{8}\mathbb{E}\|Z\|^p,
\end{aligned}$$

where the second inequality follows by the assumptions of Case 2 and the definition of Φ and the last one by Lemma 14. Hence, Lemma 15 yields

$$\mathbb{E}\|Y + Z\|^p \geq \mathbb{E}\|Y\|^p + \frac{1}{2}\mathbb{E}\|Z\|^p.$$

Thus

$$\mathbb{E}\left\|\sum_{i=0}^{n+1} v_iR_i\right\|^p \mathbb{1}_{A_k} \geq \frac{1}{2}\mathbb{E}\left\|\sum_{i=0}^{k-1} v_iR_i\right\|^p \mathbb{1}_{A_k} + \mathbb{E}\left\|\sum_{i=k}^{n+1} v_iR_i\right\|^p \mathbb{1}_{A_k}. \quad (33)$$

Using Lemma 12 and (32) we obtain

$$\mathbb{E}\left\|\sum_{i=0}^{k-1} v_iR_i\right\|^p \mathbb{1}_{A_k} \geq \delta\|v_0\|^p\mathbb{P}(R_{2,k} \leq 4\lambda^{2k-2}) \geq \frac{\delta}{2}\|v_0\|^p.$$

Since $\varepsilon_0 \leq \frac{1}{4}$ and $\varepsilon_0 \leq \delta/8$ it follows that

$$\frac{1}{2}\mathbb{E}\left\|\sum_{i=0}^{k-1} v_iR_i\right\|^p \mathbb{1}_{A_k} \geq \varepsilon_0\|v_0\|^p + \varepsilon_0\mathbb{E}\left\|\sum_{i=0}^{k-1} v_iR_i\right\|^p \mathbb{1}_{A_k}. \quad (34)$$

By the induction assumption we obtain

$$\mathbb{E}\left\|\sum_{i=k}^{n+1} v_iR_i\right\|^p \mathbb{1}_{A_k} \geq \varepsilon_0 \mathbb{E}\|v_kR_k\|^p \mathbb{1}_{A_k} + \sum_{i=k+1}^{n+1}\left(\frac{\varepsilon_1}{k} - c_{i-k}\right)\mathbb{E}\|v_iR_k\|^p \mathbb{1}_{A_k}. \tag{35}$$

Combining (33), (34) and (35) we arrive at

$$\begin{aligned}
\mathbb{E}\left\|\sum_{i=0}^{n+1} v_iR_i\right\|^p \mathbb{1}_{A_k} &\geq \varepsilon_0\|v_0\|^p + \varepsilon_0\mathbb{E}\left\|\sum_{i=0}^{k-1} v_iR_i\right\|^p \mathbb{1}_{A_k} + \varepsilon_0\mathbb{E}\|v_kR_k\|^p\mathbb{1}_{A_k} \\
&\quad + \sum_{i=k+1}^{n+1}\left(\frac{\varepsilon_1}{k} - c_{i-k}\right)\mathbb{E}\|v_iR_k\|^p\mathbb{1}_{A_k} \\
&\geq \varepsilon_0\|v_0\|^p + \varepsilon_0\mathbb{E}\left\|\sum_{i=0}^{k} v_iR_i\right\|^p \mathbb{1}_{A_k} + \sum_{i=k+1}^{n+1}\left(\frac{\varepsilon_1}{k} - c_{i-k}\right)\mathbb{E}\|v_iR_k\|^p\mathbb{1}_{A_k}.
\end{aligned}$$

Combining this inequality with (31) yields

$$\begin{aligned}
\mathbb{E}\left\|\sum_{i=0}^{n+1} v_iR_i\right\|^p &\geq \varepsilon_0\|v_0\|^p + \varepsilon_0\mathbb{E}\left\|\sum_{i=0}^{k} v_iR_i\right\|^p + \sum_{i=k+1}^{n+1}\left(\frac{\varepsilon_1}{k} - c_{i-k}\right)\mathbb{E}\|v_iR_k\|^p \\
&\geq \varepsilon_0\|v_0\|^p + \frac{\varepsilon_1}{k}\sum_{i=1}^{k}\|v_i\|^p + \sum_{i=k+1}^{n+1}\left(\frac{\varepsilon_1}{k} - c_{i-k}\right)\|v_i\|^p \\
&\geq \varepsilon_0\|v_0\|^p + \sum_{i=1}^{n+1}\left(\frac{\varepsilon_1}{k} - c_i\right)\|v_i\|^p,
\end{aligned}$$

where in the second inequality we used Lemma 13. □

We are now ready to establish the lower L_p-bound for $p \leq 1$.

Proof of the lower bound in Theorem 2. To show the lower bound let us choose k such that

$$k\lambda^{2k-2} \leq \frac{\delta^3(1-\lambda)^2}{2^{12}A}.$$

Then

$$c_i \leq \Phi\frac{\lambda^k}{1-\lambda} = \frac{2^8A\lambda^{2k-2}}{(1-\lambda)^2} \leq \frac{\varepsilon_1}{2k}.$$

Therefore, Proposition 16 implies

$$\mathbb{E}\left\|\sum_{i=0}^{n} v_iR_i\right\|^p \geq \frac{\delta}{8}\|v_0\|^p + \frac{\delta^3}{16k}\sum_{i=1}^{n}\|v_i\|^p \geq \frac{\delta^3}{16k}\sum_{i=0}^{n}\|v_i\|^p.$$

□

4 Upper bounds

The upper bound in Theorem 2 immediately follows by the inequality $(a+b)^p \le a^p + b^p$, $a, b \ge 0$, $p \in (0,1]$. To get the upper bound in Theorem 3 we prove the following slightly more general result.

Proposition 17. *Let $p > 0$ and $X_1, X_2, \ldots$ be independent random variables such that $\mathbb{E}|X_i|^p < \infty$ for all i and*

$$\forall_{1\le k<\lceil p\rceil}\ \exists_{\lambda_k<1}\ \forall_i\ (\mathbb{E}|X_i|^{p-k})^{1/(p-k)} \le \lambda_k (\mathbb{E}|X_i|^{p-k+1})^{1/(p-k+1)}. \tag{36}$$

Then for any vectors $v_0, v_1, \ldots, v_n$ in a normed space $(F, \|\ \|)$ we have

$$\mathbb{E}\left\|\sum_{i=0}^n v_i R_i\right\|^p \le C(p) \sum_{i=0}^n \|v_i\|^p \mathbb{E}|R_i|^p, \tag{37}$$

where $C(p) = 1$ for $p \le 1$ and for $p \ge 1$,

$$C(p) = 2^p \left(1 + C(p-1)\frac{\lambda_1^{p-1}}{1-\lambda_1^{p-1}}\right) \le 2^p \frac{C(p-1)}{1-\lambda_1^{p-1}}.$$

Proof. We have $\|\sum_{i=0}^n v_i R_i\| \le \sum_{i=0}^n \|v_i\||R_i|$ and $|R_i| = \prod_{j=1}^i |X_j|$, so it is enough to consider the case when $F = \mathbb{R}$, $v_k \ge 0$ and variables X_j are nonnegative. Since it is only a matter of normalization we may also assume that $\mathbb{E}X_i^p = 1$ for all i.

We proceed by induction on $m := \lceil p \rceil$. If $m = 1$, i.e. $0 < p \le 1$ then the assertion easily follows, since $(x+y)^p \le x^p + y^p$, $x, y \ge 0$.

Suppose that $m > 1$ and (37) holds in the case $p \le m$. Take p such that $m < p \le m+1$. Observe that

$$(x+y)^p \le x^p + 2^p(yx^{p-1} + y^p) \quad \text{for } x, y \ge 0. \tag{38}$$

Indeed, either $x \le y$ and then $(x+y)^p \le 2^p y^p$, or $0 \le y < x$ and then by the convexity of x^p, $((x+y)^p - x^p)/y \le ((2x)^p - x^p)/x = (2^p - 1)x^{p-1}$.

We have by (38)

$$\mathbb{E}\left|\sum_{i=0}^n v_i R_i\right|^p \le \mathbb{E}\left|\sum_{i=1}^n v_i R_i\right|^p + 2^p \left(v_0 \mathbb{E}\left|\sum_{i=1}^n v_i R_i\right|^{p-1} + v_0^p\right).$$

Iterating this inequality we get

$$\mathbb{E}\left|\sum_{i=0}^n v_i R_i\right|^p \le v_n^p \mathbb{E}R_n^p + 2^p \left(\sum_{k=0}^{n-1} v_k \mathbb{E}R_k \left(\sum_{i=k+1}^n v_i R_i\right)^{p-1} + \sum_{i=0}^{n-1} v_i^p \mathbb{E}R_i^p\right).$$

However, $\mathbb{E}R_k(\sum_{i=k+1}^n v_iR_i)^{p-1} = \mathbb{E}R_k^p\mathbb{E}(\sum_{i=k+1}^n v_iR_{k+1,i})^{p-1}$ and $\mathbb{E}R_k^p = \prod_{j=1}^k \mathbb{E}X_j^p = 1$. Hence

$$\mathbb{E}\left|\sum_{i=0}^n v_iR_i\right|^p \le 2^p\sum_{i=0}^n v_i^p + 2^p\sum_{k=0}^{n-1} v_k\mathbb{E}\left(\sum_{i=k+1}^n v_iR_{k+1,i}\right)^{p-1}.$$

The induction assumption yields

$$\begin{aligned}\mathbb{E}\left(\sum_{i=k+1}^n v_iR_{k+1,i}\right)^{p-1} &\le C(p-1)\sum_{i=k+1}^n v_i^{p-1}\mathbb{E}R_{k+1,i}^{p-1} = C(p-1)\sum_{i=k+1}^n v_i^{p-1}\prod_{j=k+1}^i \mathbb{E}X_j^{p-1}\\ &\le C(p-1)\sum_{i=k+1}^n v_i^{p-1}\lambda_1^{(p-1)(i-k)},\end{aligned}$$

where the last inequality follows by (36). To finish the proof we observe that

$$\begin{aligned}\sum_{k=0}^{n-1} v_k\sum_{i=k+1}^n v_i^{p-1}\lambda_1^{(p-1)(i-k)} &\le \sum_{0\le k<i\le n}\left(\frac{1}{p}v_k^p + \frac{p-1}{p}v_i^p\right)\lambda_1^{(p-1)(i-k)}\\ &\le \sum_{i=0}^n v_i^p\sum_{j=1}^\infty \lambda_1^{(p-1)j} = \frac{\lambda_1^{p-1}}{1-\lambda_1^{p-1}}\sum_{i=0}^n v_i^p.\end{aligned}$$

□

Remark. It is not hard to show by induction on $\lceil p\rceil$ that

$$C(p) \le 2^{\frac{p(p+1)}{2}}\prod_{1\le j\le\lceil p\rceil-1}\frac{1}{1-\lambda_j^{p-j}}.$$

5 Stochastic recursions

The proof of Theorem 6 is only a slight modification of the proof of Theorem 1. Normalizing we may always assume $\mathbb{E}X^p = 1$. The upper bound follows as in the proof of Proposition 17 (see more details below). To show the lower bound we consider two cases:

$$\text{There are } w,u\in F \text{ such that } w+B+Xu=0 \text{ a.e.} \tag{C1}$$

or

$$\mathbb{P}(w+B+Xu=0)<1 \text{ for every } w,u\in F. \tag{C2}$$

In case (C1) we get

$$\begin{aligned}\sum_{i=1}^n R_{i-1}B_i &= \sum_{i=1}^n R_{i-1}(-w-X_iu) = -\sum_{i=1}^n R_{i-1}w - \sum_{i=1}^n R_iu\\ &= -\sum_{i=1}^n R_{i-1}(w+u) + u - R_nu.\end{aligned}$$

Notice that

$$\mathbb{E}\left\|\sum_{i=1}^{n} R_{i-1}(w+u)\right\|^p = \|w+u\|^p \mathbb{E}\left|\sum_{i=1}^{n} R_{i-1}\right|^p \geq c_{p,X} n \|w+u\|^p,$$

where the last inequality follows by Theorem 1 with $F = \mathbb{R}$ and $v_i = 1$. Assumption (12) implies $w + u \neq 0$. Moreover,

$$\mathbb{E}\|u - R_n u\|^p \leq 2^p \|u\|^p (1 + \mathbb{E}R_n^p) = 2^{p+1}\|u\|^p.$$

Hence for $n \geq n_0 = n_0(X, B)$ and $c = c(p, X, B) = \frac{1}{2^{p+1}} c_{p,X} \|w+u\|^p$,

$$\mathbb{E}\left\|\sum_{i=1}^{n} R_{i-1}B_i\right\|^p = \mathbb{E}\left\|\sum_{i=1}^{n} R_{i-1}(w+u) - (u - R_n u)\right\|^p \geq cn.$$

To get the lower bound in (13) for $1 \leq n < n_0$ we observe that

$$\begin{aligned} cnn_0 &\leq \mathbb{E}\left\|\sum_{i=1}^{nn_0} R_{i-1}B_i\right\|^p = \mathbb{E}\left\|\sum_{k=0}^{n_0-1}\sum_{i=kn+1}^{(k+1)n} R_{i-1}B_i\right\|^p \\ &\leq n_0^p \sum_{k=0}^{n_0-1} \mathbb{E}\left\|\sum_{i=kn+1}^{(k+1)n} R_{i-1}B_i\right\|^p = n_0^p \sum_{k=0}^{n_0-1} \mathbb{E}R_{kn}^p \mathbb{E}\left\|\sum_{i=kn+1}^{(k+1)n} R_{kn+1,i-1}B_i\right\|^p \\ &= n_0^p n_0 \mathbb{E}\left\|\sum_{i=1}^{n} R_{i-1}B_i\right\|^p, \end{aligned}$$

where the last equality follows since $\sum_{i=kn+1}^{(k+1)n} R_{kn+1,i-1}B_i$ has the same distribution as $\sum_{i=1}^{n} R_{i-1}B_i$.

It is worth mentioning here that the estimate $\mathbb{E}\|\sum_{i=1}^{n} R_{i-1}\|^p \geq cn$ was first observed in [4] under the *Goldie-Kesten conditions*. In fact, a stronger statement was proved there: $\lim_{n\to\infty} \frac{1}{n}\mathbb{E}\|\sum_{i=1}^{n} R_{i-1}\|^p$ exists and it is strictly positive. Note also that if $u = -w$, i.e. assumption (12) does not hold then

$$\mathbb{E}\left\|\sum_{i=1}^{n} R_{i-1}B_i\right\|^p = \mathbb{E}\|u - R_n u\|^p \leq 2^{p+1}\|u\|^p$$

and the lower bound in (13) cannot hold for large n.

In the sequel to derive the lower bound it is enough to consider case (C2). The following lemma is then a counterpart of Lemmas 7 and 12.

Lemma 18. *Suppose that X is a nonnegative, nondegenerate r.v., B is a random vector with values in a separable Banach space F, $\mathbb{E}|X|^p, \mathbb{E}\|B\|^p < \infty$ and for any $u, w \in F$, $\mathbb{P}(B + Xu = w) < 1$. Then there exist constants $A < \infty$ and $\delta > 0$ depending only on the distribution of (B, X) and p such that*

$$\mathbb{E}\|w + B + Xu\|^p \mathbb{1}_{\{X \le A\}} \ge \delta \max\{\|w\|^p, \|u\|^p, \mathbb{E}\|B\|^p\}.$$

Proof. By δ_1 and δ_2 in the sequel we will denote positive constants depending only on the distribution of (B, X) and p. Lemmas 7 and 12 yield

$$\mathbb{E}\|w + Xu\|^p \ge \delta_1 \max\{\|w\|^p, \|u\|^p\} \quad \text{for any } w, u \in F.$$

Since $\|u_1 + u_2\|^p \le 2^p(\|u_1\|^p + \|u_2\|^p)$ for any $u_1, u_2 \in F$, we get

$$\mathbb{E}\|w + B + Xu\|^p \ge 2^{-p}\mathbb{E}\|w + Xu\|^p - \mathbb{E}\|B\|^p \ge 2^{-p-1}\delta_1 \max\{\|w\|^p, \|u\|^p, \mathbb{E}\|B\|^p\}.$$

provided that $\max\{\|w\|^p, \|u\|^p\} \ge M := 2^{p+1}\max\{1, \delta_1^{-1}\}\mathbb{E}\|B\|^p$. Let

$$\alpha := \inf\left\{\mathbb{E}\|w + B + Xu\|^p \colon\ \max\{\|w\|^p, \|u\|^p\} \le M\right\}.$$

First we observe that $\alpha > 0$. Indeed, assume that $\alpha = 0$, then there exist sequences (u_n), (w_n) in F such that $\|u_n\|^p \le M$, $\|w_n\|^p \le M$ and $\mathbb{E}\|w_n + B + Xu_n\|^p \to 0$. We have

$$\begin{aligned}\mathbb{E}\|w_n + B + Xu_n\|^p + \mathbb{E}\|w_m + B{+}Xu_m\|^p& \\ &\ge 2^{-p}\mathbb{E}\|(w_n + B + Xu_n) - (w_m + B + Xu_m)\|^p \\ &\ge 2^{-p}\delta_1 \max\{\|w_n - w_m\|^p, \|u_n - u_m\|^p\}.\end{aligned}$$

Thus both sequences (u_n) and (w_n) satisfy the Cauchy condition, hence they are convergent, respectively to u and w. But then $\mathbb{E}\|w + B + Xu\|^p = \lim_n \mathbb{E}\|w_n + B + Xu_n\|^p = 0$, which contradicts our assumptions.

Therefore $\alpha > 0$ and for $\max\{\|w\|^p, \|u\|^p\} \le M$ we get

$$\mathbb{E}\|w + B + Xu\|^p \ge \alpha \ge \alpha \max\left\{\frac{1}{M}\|w\|^p, \frac{1}{M}\|u\|^p, \frac{1}{\mathbb{E}\|B\|^p}\mathbb{E}\|B\|^p\right\}.$$

This way we showed that

$$\mathbb{E}\|w + B + Xu\|^p \ge \delta_2 \max\{\|w\|^p, \|u\|^p, \mathbb{E}\|B\|^p\} \quad \text{for any } w, u \in F.$$

To finish the proof it is enough to note that

$$\mathbb{E}\|w{+}B{+}Xu\|^p \mathbb{1}_{\{X>A\}} \le 3^p\mathbb{E}(\|w\|^p{+}\|B\|^p{+}\|u\|^p)\mathbb{1}_{\{X>A\}} \le \frac{\delta_2}{2}\max\{\|w\|^p, \|u\|^p, \mathbb{E}\|B\|^p\}$$

provided that A is large enough. □

For the rest of the proof of the lower bound in (13) we do not need to assume that (X_i, B_i) are i.i.d, but we need uniformity in Lemma 18, i.e. the condition

$$\exists_{\delta>0,A<\infty} \forall_i \forall_{w,u\in F} \mathbb{E}\|w+B_i+X_i u\|^p \mathbb{1}_{\{X_i^p\le A\mathbb{E}X_i^p\}} \ge \delta \max\{\mathbb{E}\|B_i\|^p, \|w\|^p, \|u\|^p\mathbb{E}X_i^p\}. \quad (39)$$

More precisely, the following theorems hold.

Theorem 19. *Let $0 < p \le 1$ and let $(X_1, B_1), (X_2, B_2)... \in \mathbb{R}^+ \times F$ be a sequence of independent random variables such that $\mathbb{E}\|B_i\|^p, \mathbb{E}X_i^p < \infty$. Suppose that conditions (3) and (39) are satisfied. Then there is a constant $c(p, \lambda, \delta, A)$ such that for every n,*

$$c(p,\lambda,\delta,A)\sum_{i=1}^n (\mathbb{E}R_{i-1}^p)\mathbb{E}\|B_i\|^p \le \mathbb{E}\left\|\sum_{i=1}^n R_{i-1}B_i\right\|^p \le \sum_{i=1}^n (\mathbb{E}R_{i-1}^p)\mathbb{E}\|B_i\|^p. \quad (40)$$

Theorem 20. *Let $p > 1$ and let $(X_1, B_1), (X_2, B_2)... \in \mathbb{R}^+\times F$ be a sequence of independent random variables such that $\mathbb{E}\|B_i\|^p, \mathbb{E}X_i^p < \infty$. Suppose that conditions (6), (7) and (39) are satisfied. Then there are constants $c = c(p, q, \lambda, \delta, A), C(p, \lambda_1, \dots \lambda_{\lceil p\rceil-1})$ such that for every n,*

$$c(p,\lambda,\delta,A)\sum_{i=1}^n (\mathbb{E}R_{i-1}^p)\mathbb{E}\|B_i\|^p \le \mathbb{E}\left\|\sum_{i=1}^n R_{i-1}B_i\right\|^p \le C(p,\lambda_1,\dots\lambda_{\lceil p\rceil-1})\sum_{i=1}^n (\mathbb{E}R_{i-1}^p)\mathbb{E}\|B_i\|^p. \quad (41)$$

Since it is only a matter of normalization we may and will assume that $\mathbb{E}X_i^p = 1$.

First we prove the upper bound in (41). Proceeding by induction as in the proof of Proposition 17 we get

$$\begin{aligned}\mathbb{E}\left\|\sum_{i=1}^n R_{i-1}B_i\right\|^p &\le \mathbb{E}\left\|\sum_{i=2}^n R_{i-1}B_i\right\|^p + 2^p\left(\mathbb{E}\|B_1\|\left(\sum_{i=2}^n R_{i-1}\|B_i\|\right)^{p-1} + \mathbb{E}\|B_1\|^p\right)\\ &= \mathbb{E}\left\|\sum_{i=2}^n R_{i-1}B_i\right\|^p + 2^p\left(\mathbb{E}\|B_1\|X_1^{p-1}\left(\sum_{i=2}^n R_{2,i-1}\|B_i\|\right)^{p-1} + \mathbb{E}\|B_1\|^p\right).\end{aligned}$$

Iterating this inequality we obtain

$$\begin{aligned}\mathbb{E}\left\|\sum_{i=1}^n R_{i-1}B_i\right\|^p &\le \mathbb{E}\|B_n\|^p + 2^p\sum_{k=1}^{n-1}\mathbb{E}\|B_k\|R_k^{p-1}\left(\sum_{i=k+1}^n R_{k+1,i-1}\|B_i\|\right)^{p-1} + 2^p\sum_{i=1}^{n-1}\mathbb{E}\|B_i\|^p\\ &\le 2^p\sum_{i=1}^n \mathbb{E}\|B_i\|^p + 2^p\sum_{k=1}^{n-1}\mathbb{E}\|B_k\|X_k^{p-1}\mathbb{E}\left(\sum_{i=k+1}^n R_{k+1,i-1}\|B_i\|\right)^{p-1}.\end{aligned}$$

By the induction assumption

$$\mathbb{E}\left(\sum_{i=k+1}^{n} R_{k+1,i-1}\|B_i\|\right)^{p-1} \le C(p-1)\sum_{i=k+1}^{n}(\mathbb{E}R_{k+1,i-1}^{p-1})\mathbb{E}\|B_i\|^{p-1}$$
$$\le C(p-1)\sum_{i=k+1}^{n}\lambda_1^{(i-1-k)(p-1)}\mathbb{E}\|B_i\|^{p-1}.$$

Hence

$$\mathbb{E}\left\|\sum_{i=1}^{n} R_{i-1}B_i\right\|^p \le 2^p\sum_{i=1}^{n}\|B_i\|^p + 2^pC(p-1)\sum_{k=1}^{n-1}\sum_{i=k+1}^{n}\lambda_1^{(i-1-k)(p-1)}\mathbb{E}\|B_k\|X_k^{p-1}\|B_i\|^{p-1}.$$

To finish the proof of the upper bound we observe that for $k < i$,

$$\mathbb{E}\|B_k\|X_k^{p-1}\|B_i\|^{p-1} \le \frac{1}{p}\mathbb{E}(\|B_k\|^p + (p-1)X_k^p\|B_i\|^p) = \frac{1}{p}(\mathbb{E}\|B_k\|^p + (p-1)\mathbb{E}\|B_i\|^p).$$

Therefore

$$\sum_{k=1}^{n-1}\sum_{i=k+1}^{n}\lambda_1^{(i-1-k)(p-1)}\mathbb{E}\|B_k\|X_k^{p-1}\|B_i\|^{p-1}$$
$$\le \sum_{k=1}^{n-1}\sum_{i=k+1}^{n}\lambda_1^{(i-1-k)(p-1)}\left(\frac{1}{p}\mathbb{E}\|B_k\|^p + \frac{p-1}{p}\mathbb{E}\|B_i\|^p\right) \le \frac{1}{1-\lambda_1^{p-1}}\sum_{i=1}^{n}\mathbb{E}\|B_i\|^p$$

and the conclusion follows.

To prove the lower bounds in (40) and (41) we follow closely arguments of Sections 2 and 3 making use of (39) whenever Lemma 7 or Lemma 12 are used. For instance, to obtain the estimate

$$\mathbb{E}\left\|w + \sum_{i=1}^{n} R_{i-1}B_i\right\|^p \ge \delta \max_{1\le j\le n}\mathbb{E}\|B_j\|^p \ge \frac{\delta}{n}\sum_{j=1}^{n}\mathbb{E}\|B_j\|^p \tag{42}$$

we proceed as follows. For $1 \le j \le n$ we have

$$\mathbb{E}\left\|w + \sum_{i=1}^{n} R_{i-1}B_i\right\|^p \ge \mathbb{E}\left\|w + \sum_{i=1}^{n} R_{i-1}B_i\right\|^p \mathbb{1}_{\{R_{j-1}>0\}} = \mathbb{E}R_{j-1}^p\|Y_j + B_j + X_jZ_j\|^p,$$

where

$$Y_j := \left(w + \sum_{i=1}^{j-1} R_{i-1}B_i\right)\frac{1}{R_{j-1}}\mathbb{1}_{\{R_{j-1}>0\}} \quad \text{and} \quad Z_j := \sum_{i=j+1}^{n} R_{j+1,i-1}B_i.$$

Since variables R_{j-1}, Y_j and Z_j are independent of (X_j, B_j) condition (39) yields

$$\mathbb{E}\left\| w + \sum_{i=1}^{n} R_{i-1}B_i \right\|^p \geq \delta \mathbb{E} R_{j-1}^p \mathbb{E}\|B_j\|^p = \delta \mathbb{E}\|B_j\|^p.$$

Similar argument used for $j = 1$ yields

$$\mathbb{E}\left\| w + \sum_{i=1}^{n} R_{i-1}B_i \right\|^p \geq \delta \|w\|^p. \tag{43}$$

For the rest let us concentrate on the case $p \leq 1$ presenting only the parts of the argument that are specific for the setting of Theorem 19. If $p > 1$ the argument is completely analogous. In this situation Lemma 14 holds with the same proof.

Lemma 21. *Suppose the assumptions of Theorem 19 are satisfied. Then for $t > 0$,*

$$\mathbb{P}\left(\left\| \sum_{i=1}^{n} X_1 \dots X_{i-1} B_i \right\|^p \geq \frac{t}{1-\lambda} \sum_{i=1}^{n} \lambda^{i-1} \mathbb{E}\|B_i\|^p \right) \leq t^{-1/2}. \tag{44}$$

The main proposition (analogous to Proposition 16) can be formulated as follows.

Proposition 22. *Suppose that the assumptions of Theorem 19 are satisfied and $\mathbb{E}X_i^p = 1$ for all i. Then for any $w \in F$ and $k = 1, 2, \dots$ we have*

$$\mathbb{E}\left\| w + \sum_{i=1}^{n} R_{i-1}B_i \right\|^p \geq \varepsilon_0 \|w\|^p + \sum_{i=1}^{n} \left(\frac{\varepsilon_1}{k} - c_i \right) \mathbb{E}\|B_i\|^p,$$

where $\varepsilon_0 = \delta/8$, $\varepsilon_1 = \delta\varepsilon_0$,

$$c_i = 0 \textit{ for } 1 \leq i \leq k-1, \quad c_i = \Phi \sum_{j=k}^{i} \lambda^{j-1}, i \geq k \quad \textit{and} \quad \Phi = \frac{2^8 A}{(1-\lambda)} \lambda^{k-2}.$$

Proof. For $n \leq k$ the assertion follows by (42) and (43). For $n \geq k$ we proceed by induction. To simplify the notation let for $k = 1, 2, \dots$ and $w \in F$,

$$S_{k,n}(w) := w + \sum_{i=k}^{n} R_{k,i-1}B_i \quad \text{and} \quad S_n(w) := S_{1,n}(w) = w + \sum_{i=k}^{n} R_{i-1}B_i.$$

Observe that the random variable $S_{k,n}(w)$ is independent of $(X_i, B_i)_{i \leq k-1}$.

As in the proof of Proposition 16 we consider two cases. First assume that

$$\varepsilon_0 \|w\|^p \leq \Phi \sum_{i=k}^{n+1} \lambda^{i-1} \mathbb{E}\|B_i\|^p. \tag{45}$$

We have

$$\mathbb{E}\|S_{n+1}(w)\|^p = \mathbb{E}X_1^p\|S_{2,n+1}(w')\|^p\mathbb{1}_{\{X_1>0\}} + \mathbb{E}\|w+B_1\|^p\mathbb{1}_{\{X_1=0\}}, \tag{46}$$

where $w' = X_1^{-1}(w+B_1)\mathbb{1}_{\{X_1>0\}}$. Hence by the induction assumption (used conditionally on (X_1, B_1)) we get

$$\begin{aligned}
\mathbb{E}\|S_{n+1}(w)\|^p &\geq \mathbb{E}X_1^p\left(\varepsilon_0\|w'\|^p + \sum_{i=2}^{n+1}\left(\frac{\varepsilon_1}{k} - c_{i-1}\right)\mathbb{E}\|B_i\|^p\right)\mathbb{1}_{\{X_1>0\}} + \mathbb{E}\|w+B_1\|^p\mathbb{1}_{\{X_1=0\}} \\
&= \varepsilon_0\mathbb{E}\|w+B_1\|^p\mathbb{1}_{\{X_1>0\}} + \sum_{i=2}^{n+1}\left(\frac{\varepsilon_1}{k} - c_{i-1}\right)\mathbb{E}X_1^p\|B_i\|^p\mathbb{1}_{\{X_1>0\}} \\
&\quad + \mathbb{E}\|w+B_1\|^p\mathbb{1}_{\{X_1=0\}} \\
&\geq \varepsilon_0\mathbb{E}\|w+B_1\|^p + \sum_{i=2}^{n+1}\left(\frac{\varepsilon_1}{k} - c_{i-1}\right)\mathbb{E}\|B_i\|^p \\
&\geq \varepsilon_1\mathbb{E}\|B_1\|^p + \sum_{i=2}^{n+1}\left(\frac{\varepsilon_1}{k} - c_{i-1}\right)\mathbb{E}\|B_i\|^p + \varepsilon_0\|w\|^p - \Phi\sum_{i=k}^{n+1}\lambda^{i-1}\mathbb{E}\|B_i\|^p \\
&= \varepsilon_0\|w\|^p + \varepsilon_1\mathbb{E}\|B_1\|^p + \sum_{i=2}^{n+1}\left(\frac{\varepsilon_1}{k} - c_i\right)\mathbb{E}\|B_i\|^p,
\end{aligned}$$

where we used independence of X_1 and B_i for $i \geq 2$, normalization $\mathbb{E}X_1^p\mathbb{1}_{\{X_1>0\}} = \mathbb{E}X_1^p = 1$ and inequalities (42) and (45).

Now suppose that

$$\varepsilon_0\|w\|^p > \Phi\sum_{i=k}^{n+1}\lambda^{i-1}\mathbb{E}\|B_i\|^p \tag{47}$$

and let

$$U_k := \{X_1^p \leq A, R_{2,k}^p \leq 4\lambda^{2k-2}\}.$$

We have

$$\begin{aligned}
\mathbb{E}\|S_{n+1}(w)\|^p\mathbb{1}_{\Omega\setminus U_k} &= \mathbb{E}R_k^p\|S'_{k+1,n+1}(w')\|^p\mathbb{1}_{\Omega\setminus U_k}\mathbb{1}_{\{R_k>0\}} \\
&\quad + \mathbb{E}\|w + \sum_{i=1}^{k}R_{i-1}B_i\|^p\mathbb{1}_{\Omega\setminus U_k}\mathbb{1}_{\{R_k=0\}},
\end{aligned}$$

where $w' = (R_k)^{-1}(w + \sum_{i=1}^k R_{i-1}B_i)\mathbb{1}_{\{R_k>0\}}$. Hence by the induction assumption

$$
\begin{aligned}
&\mathbb{E}\|S_{n+1}(w)\|^p \mathbb{1}_{\Omega\setminus U_k} \\
&\quad\geq \mathbb{E}R_k^p \left(\varepsilon_0\|w'\|^p + \sum_{i=k+1}^{n+1} \left(\frac{\varepsilon_1}{k} - c_{i-k}\right) \mathbb{E}\|B_i\|^p \right) \mathbb{1}_{\Omega\setminus U_k}\mathbb{1}_{\{R_k>0\}} \\
&\qquad + \mathbb{E}\left\| w + \sum_{i=1}^k R_{i-1}B_i \right\|^p \mathbb{1}_{\Omega\setminus U_k}\mathbb{1}_{\{R_k=0\}} \\
&\quad\geq \varepsilon_0 \mathbb{E}\left\| w + \sum_{i=1}^k R_{i-1}B_i \right\|^p \mathbb{1}_{\Omega\setminus U_k}\mathbb{1}_{\{R_k>0\}} + \mathbb{E}R_k^p \sum_{i=k+1}^{n+1} \left(\frac{\varepsilon_1}{k} - c_{i-k}\right) \mathbb{E}\|B_i\|^p \mathbb{1}_{\Omega\setminus U_k}\mathbb{1}_{\{R_k>0\}} \\
&\qquad + \mathbb{E}\left\| w + \sum_{i=1}^k R_{i-1}B_i \right\|^p \mathbb{1}_{\Omega\setminus U_k}\mathbb{1}_{\{R_k=0\}} \\
&\quad\geq \varepsilon_0 \mathbb{E}\left\| w + \sum_{i=1}^k R_{i-1}B_i \right\|^p \mathbb{1}_{\Omega\setminus U_k} + \mathbb{E}R_k^p \sum_{i=k+1}^{n+1} \left(\frac{\varepsilon_1}{k} - c_{i-k}\right) \mathbb{E}\|B_i\|^p \mathbb{1}_{\Omega\setminus U_k} \\
&\quad\geq \varepsilon_0 \mathbb{E}\left\| w + \sum_{i=1}^k R_{i-1}B_i \right\|^p \mathbb{1}_{\Omega\setminus U_k} + \sum_{i=k+1}^{n+1} \left(\frac{\varepsilon_1}{k} - c_{i-k}\right) \mathbb{E}\|R_kB_i\|^p \mathbb{1}_{\Omega\setminus U_k}.
\end{aligned}
$$

To finish the proof we define (Z, Y, Y') as the random variable

$$
\left(w + \sum_{i=1}^k X_k \dots X_{i-1}B_i, \sum_{i=k+1}^{n+1} X_1 \dots X_{i-1}B_i, \sum_{i=k+1}^{n+1} X_{k+1} \dots X_{i-1}B_i \right)
$$

conditioned on U_k and we proceed as in the proof of Proposition 16. □

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