# ON A LOOMIS-WHITNEY TYPE INEQUALITY FOR PERMUTATIONALLY INVARIANT UNCONDITIONAL CONVEX BODIES

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ABSTRACT. For a permutationally invariant unconditional convex body $K$ in $\mathbb{R}^n$ we define a finite sequence $(K_j)_{j=1}^n$ of projections of the body $K$ to the space spanned by first $j$ vectors of the standard basis of $\mathbb{R}^n$. We prove that the sequence of volumes $(|K_j|)_{j=1}^n$ is log-concave.

## 1. INTRODUCTION

The main interest in convex geometry is the examination of sections and projections of sets. Some introduction can be found in a monograph by Gardner, [4]. We are interested in a class $\mathcal{PU}_n$ of convex bodies in $\mathbb{R}^n$ which are unconditional and permutationally invariant.

Let us briefly recall some definitions. A convex body $K$ in $\mathbb{R}^n$ is called *unconditional* if for every point $(x_1, \ldots, x_n) \in K$ and every choice of signs $\epsilon_1, \ldots, \epsilon_n \in \{-1, 1\}$ the point $(\epsilon_1 x_1, \ldots, \epsilon_n x_n)$ also belongs to $K$. A convex body $K$ in $\mathbb{R}^n$ is called *permutationally invariant* if for every point $(x_1, \ldots, x_n) \in K$ and every permutation $\pi : \{1, \ldots, n\} \longrightarrow \{1, \ldots, n\}$ the point $(x_{\pi(1)}, \ldots, x_{\pi(n)})$ is also in $K$. A sequence $(a_i)_{i=1}^n$ of positive real numbers is called *log-concave* if $a_i^2 \geq a_{i-1}a_{i+1}$, for $i = 2, \ldots, n-1$.

The main result of this paper reads as follows.

**Theorem 1.** *Let $n \geq 3$ and let $K \in \mathcal{PU}_n$. For each $i = 1, \ldots, n$ we define a convex body $K_i \in \mathcal{PU}_i$ as an orthogonal projection of $K$ to the subspace $\{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_{i+1} = \ldots = x_n = 0\}$. Then the sequence of volumes $(|K_i|)_{i=1}^n$ is log-concave. In particular*

$$|K_{n-1}|^2 \geq |K_n| \cdot |K_{n-2}|. \tag{1}$$

Inequality (1) is related to the problem of negative correlation of coordinate functions on $K \in \mathcal{PU}_n$, i.e. the question whether for every $t_1, \ldots, t_n \geq 0$

$$\mu_K\left(\bigcap_{i=1}^n \{|x_i| \geq t_i\}\right) \leq \prod_{i=1}^n \mu_K\left(|x_i| \geq t_i\right), \tag{2}$$

where $\mu_K$ is normalized Lebesgue measure on $K$. Indeed, the Taylor expansion of the function $h(t) = \mu_K(|x_1| \geq t)\mu_K(|x_2| \geq t) - \mu_K(|x_1| \geq t, |x_2| \geq t)$ at $t = 0$ contains

$$\frac{1}{|K_n|^2}\left(|K_{n-1}|^2 - |K_{n-2}| \cdot |K_n|\right) t^2,$$

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cf. (1), as a leading term. The property (2), the so-called concentration hypothesis and the central limit theorem for convex bodies are closely related, see [1]. The last theorem has been recently proved by Klartag, [8].

The negative correlation property in the case of generalized Orlicz balls was originally investigated by Wojtaszczyk in [11]. A generalized Orlicz ball is a set

$$B = \left\{ (x_1, \ldots, x_n) \in \mathbb{R}^n \mid \sum_{i=1}^{n} f_i(|x_i|) \le n \right\},$$

where $f_1, \ldots, f_n$ are some Young functions (see [11] for the definition). In probabilistic terms Pilipczuk and Wojtaszczyk (see [10]) have shown that the random variable $X = (X_1, \ldots, X_n)$ uniformly distributed on $B$ satisfies the inequality

$$\mathrm{Cov}(f(|X_{i_1}|, \ldots, |X_{i_k}|), g(|X_{j_1}|, \ldots, |X_{j_l}|)) \le 0$$

for any bounded coordinate-wise increasing functions $f\colon \mathbb{R}^k \longrightarrow \mathbb{R}$, $g\colon \mathbb{R}^l \longrightarrow \mathbb{R}$ and any disjoint subsets $\{i_1, \ldots, i_k\}$ and $\{j_1, \ldots, j_l\}$ of $\{1, \ldots, n\}$. In the case of generalized isotropic Orlicz balls this result implies the inequality

$$\mathrm{Var}|X|^p \le \frac{Cp^2}{n}\mathbb{E}|X|^{2p}, \qquad p \ge 2,$$

from which some reverse Hölder inequalities can be deduced (see [3]).

One may ask about an example of a nice class of Borel probability measures on $\mathbb{R}^n$ for which the negative correlation inequality hold. Considering the example of the measure with the density

$$p(x_1, \ldots, x_n) = \exp\left(-2(n!)^{1/n} \max\{|x_1|, \ldots, |x_n|\}\right),$$

which was mentioned by Bobkov and Nazarov in a different context (see [2, Lemma 3.1]), we certainly see that the class of unconditional and permutationally invariant log-concave measures would not be the answer. Nevertheless, it remains still open whether the negative correlation of coordinate functions holds for measures uniformly distributed on the bodies from the class $\mathcal{PU}_n$.

We should remark that our inequality (1) is similar to some auxiliary result by Giannopoulos, Hartzoulaki and Paouris, see [7, Lemma 4.1]. They proved that a version of inequality (1) holds, up to the multiplicative constant $\frac{n}{2(n-1)}$, for an arbitrary convex body.

The paper is organised as follows. In Section 2 we give the proof of Theorem 1. Section 3 is devoted to some remarks. Several examples are there provided as well.

## 2. Proof of the main result

Here we deal with the proof of Theorem 1. We start with an elementary lemma.

**Lemma 1.** *Let $f\colon [0, L] \longrightarrow [0, \infty)$ be a nonincreasing concave function such that $f(0) = 1$. Then*

$$\frac{n-1}{n}\left(\int_0^L f(x)^{n-2}\mathrm{d}x\right)^2 \ge \int_0^L x f(x)^{n-2}\mathrm{d}x, \qquad n \ge 3. \tag{3}$$

*Proof.* By a linear change of a variable one can assume that $L = 1$. Since $f$ is concave and nonincreasing, we have $1 - x \leq f(x) \leq x$ for $x \in [0,1]$. Therefore, there exists a real number $\alpha \in [0,1]$ such that for $g(x) = 1 - \alpha x$ we have

$$\int_0^1 f(x)^{n-2}\mathrm{d}x = \int_0^1 g(x)^{n-2}\mathrm{d}x.$$

Clearly, we can find a number $c \in [0,1]$ such that $f(c) = g(c)$. Since $f$ is concave and $g$ is affine, we have $f(x) \geq g(x)$ for $x \in [0,c]$ and $f(x) \leq g(x)$ for $x \in [c,1]$. Hence,

$$\begin{aligned}\int_0^1 x(f(x)^{n-2} - g(x)^{n-2})\mathrm{d}x \leq& \int_0^c c(f(x)^{n-2} - g(x)^{n-2})\mathrm{d}x \\ &+ \int_c^1 c(f(x)^{n-2} - g(x)^{n-2})\mathrm{d}x \ = \ 0.\end{aligned}$$

We conclude that it suffices to prove (3) for the function $g$, which is by simple computation equivalent to

$$\begin{aligned}\frac{1}{\alpha^2 n(n-1)}\left(1-(1-\alpha)^{n-1}\right)^2 \geq \frac{1}{\alpha^2}\Bigg(&\frac{1}{n-1}\left(1-(1-\alpha)^{n-1}\right) \\ &- \frac{1}{n}\left(1-(1-\alpha)^{n}\right)\Bigg).\end{aligned}$$

To finish the proof one has to perform a short calculation and use Bernoulli's inequality.

□

**Remark 1.** A slightly more general form of this lemma appeared in [6] and, as it is pointed out in that paper, the lemma is a particular case of a result of [9, p. 182]. Only after the paper was written we heard about these references from Prof. A. Zvavitch, for whom we are thankful. Our proof differs only in a few details, yet it is provided for the convenience of the reader.

*Proof of Theorem 1.* Due to an inductive argument it is enough to prove inequality (1).

Let $g\colon \mathbb{R}^{n-1} \longrightarrow \{0,1\}$ be a characteristic function of the set $K_{n-1}$. Then, by permutational invariance and unconditionality, we have

$$|K_{n-1}| = 2^{n-1}(n-1)! \int_{x_1 \geq \ldots \geq x_{n-1} \geq 0} g(x_1, \ldots, x_{n-1})\mathrm{d}x_1 \ldots \mathrm{d}x_{n-1}, \tag{4}$$

and similarly

$$|K_{n-2}| = 2^{n-2}(n-2)! \int_{x_1 \geq \ldots \geq x_{n-2} \geq 0} g(x_1, \ldots, x_{n-2}, 0)\mathrm{d}x_1 \ldots \mathrm{d}x_{n-2}. \tag{5}$$

Moreover, permutational invariance and the definition of a projection imply

$$\mathbf{1}_{K_n}(x_1, \ldots, x_n) \leq \prod_{i=1}^{n} g(x_1, \ldots, \hat{x_i}, \ldots, x_n). \tag{6}$$

Thus

$$
\begin{aligned}
|K_n| &\le 2^n n! \int_{x_1\ge\ldots\ge x_n\ge 0} \prod_{i=1}^n g(x_1,\ldots,\hat{x_i},\ldots,x_n)\mathrm{d}x_1\ldots\mathrm{d}x_n \\
&= 2^n n! \int_{x_1\ge\ldots\ge x_n\ge 0} g(x_1,\ldots,x_{n-1})\mathrm{d}x_1\ldots\mathrm{d}x_n \qquad (7)\\
&= 2^n n! \int_{x_1\ge\ldots\ge x_{n-1}\ge 0} x_{n-1}g(x_1,\ldots,x_{n-1})\mathrm{d}x_1\ldots\mathrm{d}x_{n-1},
\end{aligned}
$$

where the first equality follows from the monotonicity of the function $g$ for nonnegative arguments with respect to each coordinate. We define a function $F\colon [0,\infty) \longrightarrow [0,\infty)$ by the equation

$$F(x) = \frac{\int_{x_1\ge\ldots\ge x_{n-2}\ge x} g(x_1,\ldots,x_{n-2},x)\mathrm{d}x_1\ldots\mathrm{d}x_{n-2}}{\int_{x_1\ge\ldots\ge x_{n-2}\ge 0} g(x_1,\ldots,x_{n-2},0)\mathrm{d}x_1\ldots\mathrm{d}x_{n-2}}.$$

One can notice that

1. $F(0)=1$.
2. The function $F$ is nonincreasing as so is the function
$$x \mapsto g(x_1,\ldots,x_{n-2},x)\mathbf{1}_{\{x_1\ge\ldots\ge x_{n-2}\ge x\}}.$$
3. The function $F^{1/(n-2)}$ is concave on its support $[0,L]$ since $F(x)$ multiplied by some constant equals the volume of the intersection of the convex set $K_{n-1}\cap\{x_1\ge\ldots\ge x_{n-1}\ge 0\}$ with the hyperplane $\{x_{n-1}=x\}$. This is a simple consequence of the Brunn-Minkowski inequality, see for instance [5, page 361].

By the definition of the function $F$ and equations (4), (5) we obtain

$$\int_0^L F(x)\mathrm{d}x = \frac{\frac{1}{2^{n-1}(n-1)!}|K_{n-1}|}{\frac{1}{2^{n-2}(n-2)!}|K_{n-2}|} = \frac{1}{2(n-1)}\cdot\frac{|K_{n-1}|}{|K_{n-2}|},$$

and using inequality (7)

$$\int_0^L xF(x)\mathrm{d}x \ge \frac{\frac{1}{2^n n!}|K_n|}{\frac{1}{2^{n-2}(n-2)!}|K_{n-2}|} = \frac{1}{2^2 n(n-1)}\cdot\frac{|K_n|}{|K_{n-2}|}.$$

Therefore it is enough to show that

$$\frac{n-1}{n}\left(\int_0^L F(x)\mathrm{d}x\right)^2 \ge \int_0^L xF(x)\mathrm{d}x.$$

This inequality follows from Lemma 1. □

## 3. Some remarks

In this section we give some remarks concerning Theorem 1.

**Remark 2.** Apart from the trivial example of the $B_\infty^n$ ball, there are many other examples of bodies for which equality in (1) is attained. Indeed, analysing the proof, we observe that for the equality in (1) the equality in Lemma 1 is needed. Therefore, the function $F^{1/(n-2)}$ has to be linear and equal to $1-x$. Taking into account the equality conditions in the Brunn-Minkowski inequality (consult [5, page 363]), this is the case if and only if the set $K_{n-1} \cap \{x_1 \geq \ldots \geq x_{n-1} \geq 0\}$ is a cone $C$ with the base $(K_{n-2} \cap \{x_1 \geq \ldots \geq x_{n-2} \geq 0\}) \times \{0\} \subset \mathbb{R}^{n-1}$ and the vertex $(z_0, \ldots, z_0) \in \mathbb{R}^{n-1}$. Thus if for a convex body $K \in \mathcal{PU}_n$ we have the equality in (1), then this body $K$ is constructed in the following manner. Take an arbitrary $K_{n-2} \in \mathcal{PU}_{n-2}$. Define the set $K_{n-1}$ as the smallest permutationally invariant unconditional body containing $C$. For $z_0$ from some interval the set $K_{n-1}$ is convex. For the characteristic function of the body $K$ we then set $\prod_{i=1}^{n} \mathbf{1}_{K_{n-1}}(x_1, \ldots, \hat{x_i}, \ldots, x_n)$.

A one more natural question to ask is when a sequence $(|K_i|)_{i=1}^n$ is geometric Bearing in mind what has been said above for $i = 2, 3, \ldots, n-1$ we find that a sequence $(|K_i|)_{i=1}^n$ is geometric if and only if

$$K = [-L, L]^n \ \cup \bigcup_{i \in \{1,\ldots,n\}, \epsilon \in \{-1,1\}} \operatorname{conv}\{\epsilon a e_i, \{x_i = \epsilon L, |x_k| \leq L, k \neq i\}\},$$

for some positive parameters $a$ and $L$ satisfying $L < a < 2L$, where $e_1, \ldots, e_n$ stand for the standard orthonormal basis in $\mathbb{R}^n$. One can easily check that $|K_i| = 2^i L^{i-1} a$.

**Remark 3.** Suppose we have a sequence of convex bodies $K_n \in \mathcal{PU}_n$, for $n \geq 1$, such that $K_n = \pi_n(K_{n+1})$, where by $\pi_n \colon \mathbb{R}^{n+1} \longrightarrow \mathbb{R}^n$ we denote the projection $\pi_n(x_1, \ldots, x_n, x_{n+1}) = (x_1, \ldots, x_n)$. Since Theorem 1 implies that the sequence $(|K_n|)_{n=1}^\infty$ is log-concave we deduce the existence of the limits

$$\lim_{n\to\infty} \frac{|K_{n+1}|}{|K_n|}, \qquad \lim_{n\to\infty} \sqrt[n]{|K_n|}.$$

We can obtain this kind of sequences as finite dimensional projections of an Orlicz ball in $\ell_\infty$.

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