RESILIENCE OF CUBE SLICING IN ℓ_p

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Abstract. Ball's celebrated cube slicing (1986) asserts that among hyperplane sections of the cube in $\mathbb{R}^n$, the central section orthogonal to $(1,1,0,\ldots,0)$ has the greatest volume. We show that the same continues to hold for slicing ℓ_p balls when $p > 10^{15}$, as well as that the same hyperplane minimizes the volume of projections of ℓ_q balls for $1 < q < 1+10^{-12}$. This extends Szarek's optimal Khinchin inequality (1976) which corresponds to $q = 1$. These results thus address the resilience of the Ball–Szarek hyperplane in the ranges $2 < p < \infty$ and $1 < q < 2$, where analysis of the extremizers has been elusive since the works of Koldobsky (1998), Barthe–Naor (2002) and Oleszkiewicz (2003).

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1. Introduction

Fix $p \in [1,\infty]$ and $n \in \mathbb{N}$. The present paper is devoted to the study of geometric parameters of the origin symmetric convex bodies

$$\mathsf{B}_p^n = \big\{x \in \mathbb{R}^n : \ \|x\|_p \le 1\big\},$$

which are the closed unit balls of the normed spaces $\ell_p^n = (\mathbb{R}^n, \|\cdot\|_p)$, where for $p \in [1,\infty)$,

$$\|x\|_p = \big(|x_1|^p + \ldots + |x_n|^p\big)^{1/p}$$

and $\|x\|_\infty = \max_{i=1,\ldots,n} |x_i|$, when $x = (x_1,\ldots,x_n) \in \mathbb{R}^n$. More specifically, we shall address the classical problem of identifying volume extremizing sections and projections of these bodies with respect to hyperplanes passing through the origin. This subject has attracted the interest of mathematicians for decades and a range of tools from probability and Fourier analysis have been employed in its study. We refer to the survey [26] for a detailed account of classical results, recent advances and further references.

1.1. Sections.

Fix $p \in [1,\infty]$, $n \in \mathbb{N}$ and consider the following question for sections of B_p^n.

Question 1. For which unit vectors a in $\mathbb{R}^n$ is the volume of $\mathsf{B}_p^n \cap a^\perp$ maximal or minimal?

This problem and its variations has been intensively studied for five decades, since Hadwiger and Hensley showed in [11, 12] that sections of the cube B_∞^n with coordinate hyperplanes $e_i^\perp$ have minimal volume. The reverse question of identifying the volume maximizing sections of the cube was answered in Ball's monumental work [1], who proved that

$$\mathrm{vol}\big(\mathsf{B}_\infty^n \cap a^\perp\big) \le \mathrm{vol}\big(\mathsf{B}_\infty^n \cap \big(\tfrac{e_1+e_2}{\sqrt{2}}\big)^\perp\big). \tag{1}$$

For $p < \infty$, the study of Question 1 was initiated by Meyer and Pajor. In [25], they extended the result of Hadwiger and Hensley by proving that sections of B_p^n with coordinate hyperplanes $e_i^\perp$ have minimal volume for any $p \ge 2$ and maximal volume when $p \in [1,2]$. In the reverse direction, they showed that when $p = 1$, the section of the cross-polytope B_1^n with the hyperplane orthogonal to $\frac{e_1+\cdots+e_n}{\sqrt{n}}$ has minimal volume, a result which was later extended to all values of $p \in [1,2]$ by Koldobsky [15] (see also [9] for a different probabilistic proof).

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In view of the aforementioned results, the only missing case in the study of Question 1 is the identification of volume maximizing sections of B_p^n when $p \in (2,\infty)$, a problem that has explicitly appeared in the literature multiple times [16, 3, 28, 19, 17, 23, 8, 26]. In [28], Oleszkiewicz made a crucial remark, showing that for $p \in (2,26)$ and n large enough the section of B_p^n with the hyperplane $\big(\frac{e_1+\cdots+e_n}{\sqrt{n}}\big)^\perp$ has in fact larger volume than the section with $\big(\frac{e_1+e_2}{\sqrt{2}}\big)^\perp$ and thus one cannot expect a Ball-type extremal for all $p>2$. In the same work, he speculated that Ball-type hyperplanes may maximize the volume of sections for *sufficiently large* values of p. The first theorem of this work provides a positive answer to Oleszkiewicz's question.

Theorem 1. *There exists $26 < p_0 < 10^{15}$ such that for every $n \in \mathbb{N}$, $p \geq p_0$ and every unit vector a in $\mathbb{R}^n$, we have*

$$\mathrm{vol}\big(\mathsf{B}_p^n \cap a^\perp\big) \leq \mathrm{vol}\big(\mathsf{B}_p^n \cap \big(\tfrac{e_1+e_2}{\sqrt{2}}\big)^\perp\big). \tag{2}$$

This is the first available result on maximal sections of B_p^n for $p \in (2,\infty)$ and *any* dimension $n \geq 3$. A general conjecture for all choices of p and n, predicting that the extremals undergo a phase transition, was proposed in [30] and [26, Conjecture 2]. Theorem 1 partially confirms it.

1.2. **Projections.** Fix $q \in [1,\infty]$, $n \in \mathbb{N}$ and consider the dual question for projections of B_q^n.

Question 2. For which unit vectors a in $\mathbb{R}^n$ is the volume of $\mathrm{Proj}_{a^\perp}\mathsf{B}_q^n$ maximal or minimal?

The current status of Question 2 is basically identical to that of Question 1. When $q=\infty$, Cauchy's projection formula shows that for every unit vector a, we have

$$\mathrm{vol}\big(\mathrm{Proj}_{a^\perp}\mathsf{B}_\infty^n\big) = \|a\|_1 \mathrm{vol}\big(\mathsf{B}_\infty^{n-1}\big), \tag{3}$$

which proves that the volume is minimized for $a = e_i$ and maximized for $a = \frac{e_1+\cdots+e_n}{\sqrt{n}}$. In the case of the cross-polytope B_1^n, similar reasoning based on Cauchy's formula (see [2]) shows that

$$\mathrm{vol}\big(\mathrm{Proj}_{a^\perp}\mathsf{B}_1^n\big) = \frac{2^{n-1}}{(n-1)!}\mathbb{E}\Big|\sum_{j=1}^n a_j\varepsilon_j\Big|, \tag{4}$$

where $\varepsilon_1, \varepsilon_2, \ldots$ is a sequence of independent symmetric ± 1 random variables. Therefore, Jensen's inequality shows that $\mathrm{vol}\big(\mathrm{Proj}_{a^\perp}\mathsf{B}_1^n\big)$ is maximal when $a = e_i$. In view of (4), identifying the volume minimizing projections of B_1^n amounts to finding the sharp constant in the classical L_1-L_2 Khinchin inequality [14] which was famously discovered by Szarek. In geometric terms, the important result of [29] asserts that $\mathrm{vol}(\mathrm{Proj}_{a^\perp}\mathsf{B}_1^n)$ is minimized for $a = \frac{e_1+e_2}{\sqrt{2}}$.

The study of Question 2 for $1<q<\infty$ was initiated by Barthe and Naor in [3]. In analogy to [25], they showed that projections of B_q^n onto coordinate hyperplanes $e_i^\perp$ have minimal volume for $q \geq 2$ and maximal volume for $q \in [1,2]$. Moreover, in the spirit of [25, 15], they proved that when $q \geq 2$, the projections of B_q^n onto the hyperplane orthogonal to $\frac{e_1+\cdots+e_n}{\sqrt{n}}$ have maximal volume (see also [18] for a different proof using the Fourier transform).

The volume minimizing hyperplane projections of B_q^n remain unknown for $q \in (1,2)$. In analogy with Oleszkiewicz's observation [28] mentioned earlier, Barthe and Naor noticed that for $q \in \big(\frac{4}{3},2\big)$, the projection of B_q^n onto the hyperplane $\big(\frac{e_1+\cdots+e_n}{\sqrt{n}}\big)^\perp$ has smaller volume than the projection onto $\big(\frac{e_1+e_2}{\sqrt{2}}\big)^\perp$ and thus one cannot expect a Szarek-type extremal for all $q \in [1,2)$. Our second theorem is the dual to Theorem 1 and addresses Question 2 for q near 1.

Theorem 2. *There exists $q_0 \in \big(1+10^{-12},\frac{4}{3}\big)$ such that for every $n \in \mathbb{N}$, $q \in [1,q_0]$ and every unit vector a in $\mathbb{R}^n$, we have*

$$\mathrm{vol}\big(\mathrm{Proj}_{a^\perp}\mathsf{B}_q^n\big) \geq \mathrm{vol}\big(\mathrm{Proj}_{\left(\frac{e_1+e_2}{\sqrt{2}}\right)^\perp}\mathsf{B}_q^n\big). \tag{5}$$

1.3. **Methods.** The delicacy of, say, Theorem 1 lies in the need to find a *universal* p_0, independent of the unit vector a and the dimension $n \in \mathbb{N}$, such that for every $p \geq p_0$,

$$\mathrm{vol}\big(\mathsf{B}_p^n \cap a^\perp\big) \leq \mathrm{vol}\big(\mathsf{B}_p^n \cap \big(\tfrac{e_1+e_2}{\sqrt{2}}\big)^\perp\big). \tag{6}$$

On the other hand, finding such a $p_0(a)$ for a *fixed* unit vector a in $\mathbb{R}^n$ is an immediate consequence of the continuity of the section function $p \mapsto \mathrm{vol}(\mathsf{B}_p^n \cap a^\perp)$, as the equality cases in Ball's inequality (1) are known to be only the vectors of the form $\frac{\pm e_i \pm e_j}{\sqrt{2}}$, where $i \neq j$.

Let $a = (a_1, \ldots, a_n)$ be a unit vector and without loss of generality assume that its coordinates are positive and ordered, i.e. $a_1 \geq a_2 \geq \ldots \geq a_n \geq 0$. Choosing p_0 uniformly for (6) to hold requires radically different arguments in the following ranges for a.

Case 1. The vector a is far from the extremizer $\frac{e_1+e_2}{\sqrt{2}}$, say $\big|a - \frac{e_1+e_2}{\sqrt{2}}\big| \geq \delta_0$ for some $\delta_0 > 0$.

The key ingredient in this range is the dimension-free stability of Ball's inequality (1) with respect to the unit vector a which has been established in recent works [6, 24] (see also Theorem 7 below for a statement with explicit constants). These works imply that, under the assumption of Case 1, there is a positive deficit in Ball's inequality. Building on the simple-minded argument based on continuity described above, one needs to reason that all functions of the form $p \mapsto \mathrm{vol}(\mathsf{B}_p^n \cap a^\perp)$ are equi-continuous at $p = \infty$ with a dimension-independent modulus. This strategy is implemented in Lemma 11 and relies on a combination of Busemann's theorem [4] with a probabilistic formula expressing the volume of sections of B_p^n as a negative moment of a sum of independent rotationally invariant random vectors in $\mathbb{R}^3$, following [13, 20, 5].

Case 2. The vector a is near the extremizer $\frac{e_1+e_2}{\sqrt{2}}$, say $\big|a - \frac{e_1+e_2}{\sqrt{2}}\big| < \delta_0$.

This range is evidently the more subtle one, as soft continuity-based arguments are deemed to fail near the equality case. In order to amend this, we introduce a novel inductive strategy. As our starting point, we express again the section function $\mathrm{vol}(\mathsf{B}_p^n \cap a^\perp)$ as a negative moment of an independent sum. After a suitable application of Jensen's inequality, we use the inductive hypothesis according to which the desired inequality holds in dimension $n-2$ and this reduces the problem to an explicit two-dimensional estimate. Quite stunningly, the resulting estimate does not hold when the unit vector a is far from the extremizer $\frac{e_1+e_2}{\sqrt{2}}$ and thus our inductive argument cannot circumvent the stability results which were crucially used in Case 1. Nevertheless, a delicate analysis allows us to deduce the technical estimate under the assumptions of Case 2 for δ_0 small enough as a function of p and p sufficiently large, thus proving Theorem 1.

The proof of Ball's inequality (1) and its stability from [6] crucially use the Fourier transform representation for the volume of sections and properties of a certain special function. However, even in Ball's original proof [1], the Fourier transform method is unable to analyze the case that the largest component a_1 of a is greater than $\frac{1}{\sqrt{2}}$, which is instead handled by an elegant geometric argument. Unfortunately, a similar geometric argument applied to B_p^n for $p < \infty$ does not yield the optimal bound (6) for a_1 slightly larger than $\frac{1}{\sqrt{2}}$, which creates the need for a different method. Surprisingly, our inductive approach outlined above does not use the Fourier transform directly, even though it uses Ball's inequality (1) and its stability as a black box. In a way, this method complements the Fourier analytic approach with a probabilistic component which permits an analysis near the extremizer.

The proof of Theorem 2 relies on a very similar strategy apart from purely technical differences. In this case, the probabilistic representation for the volume of projections is due to [3] and the stability of Szarek's inequality was obtained in [7].

2. Preliminaries

In this section we present some probabilistic representations for the volume of sections and projections of B_p^n (see also [26] and the references therein) along with some crucial technical estimates which will be used in the proofs of Theorems 1 and 2.

2.1. **Probabilistic representation of the volume of sections.** In [13], Kalton and Koldobsky discovered an elegant probabilistic representation of the volume of sections of a convex set K in $\mathbb{R}^n$ in terms of negative moments of a random vector X uniformly distributed on K. In the case of $K = \mathsf{B}_p^n$, this representation takes the following explicit form (see [5] or [26, Lemma 42]).

Lemma 3. *Fix $p \in [1, \infty)$, $n \in \mathbb{N}$ and let $Y_1, Y_2, \ldots$ be i.i.d. random variables with density $e^{-\beta_p^p |x|^p}$, where $\beta_p = 2\Gamma\big(1 + \frac{1}{p}\big)$. Then, for every unit vector a in $\mathbb{R}^n$ we have*

$$\frac{\mathrm{vol}(\mathsf{B}_p^n \cap a^\perp)}{\mathrm{vol}(\mathsf{B}_p^{n-1})} = \lim_{s \downarrow -1} \frac{1+s}{2} \mathbb{E}\Big|\sum_{j=1}^n a_j Y_j\Big|^s. \tag{7}$$

When $p = \infty$, the same identity holds with $Y_1, Y_2, \ldots$ being i.i.d. uniform on $[-1, 1]$.

Using the representation (7), we derive the following crucial formula for our analysis.

Proposition 4. *Fix $p \in [1, \infty)$ and $n \in \mathbb{N}$. Let $R_1, R_2, \ldots$ be i.i.d. positive random variables with density $\alpha_p^{-1} x^p e^{-x^p} \mathbf{1}_{x>0}$, where $\alpha_p = \frac{1}{p}\Gamma\big(1 + \frac{1}{p}\big)$ and $\xi_1, \xi_2, \ldots$ be i.i.d. random vectors uniformly distributed on the unit sphere $\mathbb{S}^2$. Then, for every unit vector a in $\mathbb{R}^n$ we have*

$$\frac{\mathrm{vol}(\mathsf{B}_p^n \cap a^\perp)}{\mathrm{vol}(\mathsf{B}_p^{n-1})} = \Gamma\Big(1 + \frac{1}{p}\Big)\ \mathbb{E}\Big|\sum_{j=1}^n a_j R_j \xi_j\Big|^{-1}, \tag{8}$$

where $|\cdot|$ denotes the Euclidean norm on the right-hand side. When $p = \infty$, the same identity holds with deterministic coefficients $R_1 = \cdots = R_n = 1$.

Proof. We shall assume that $p < \infty$ and the endpoint case follows (see also [20]). Let Y have density $e^{-\beta_p^p|x|^p}$, R have density $\alpha_p^{-1}x^p e^{-x^p}\mathbf{1}_{x>0}$ and U be uniform on $[-1, 1]$, independent of R. Then Y has the same distribution as $\beta_p^{-1} RU$. More generally, if V is a unimodal random variable with density g which is even and nonincreasing on $(0, +\infty)$, then V has the same distribution as RU, where R has density $-2rg'(r)$ on $(0, \infty)$. Indeed, for $t > 0$ we have

$$\begin{aligned}\mathbb{P}\{RU > t\} &= \mathbb{P}\Big\{U > \frac{t}{R}\Big\} = \int_0^\infty \mathbb{P}\Big\{U > \frac{t}{r}\Big\}(-2rg'(r))\mathrm{d}r = -\int_t^\infty \Big(1 - \frac{t}{r}\Big) rg'(r)\mathrm{d}r \\ &= -\int_t^\infty (r-t)\, g'(r)\mathrm{d}r = \int_t^\infty g(r)\mathrm{d}r = \mathbb{P}\{V > t\}.\end{aligned}$$

Therefore, (7) can be rewritten as

$$\frac{\mathrm{vol}(\mathsf{B}_p^n \cap a^\perp)}{\mathrm{vol}(\mathsf{B}_p^{n-1})} = \lim_{s \downarrow -1} \frac{(1+s)}{2\beta_p^s} \mathbb{E}\Big|\sum_{j=1}^n a_j R_j U_j\Big|^s. \tag{9}$$

By a result of König and Kwapień [22, Proposition 4], for every $x_1, \ldots, x_n \in \mathbb{R}$ and $s > -1$,

$$\mathbb{E}\Big|\sum_{j=1}^n x_j \xi_j\Big|^s = (1+s)\mathbb{E}\Big|\sum_{j=1}^n x_j U_j\Big|^s. \tag{10}$$

Substituting (10) in (9) conditionally on R_j and substituting the value of β_p proves (8). □

2.2. **Probabilistic representation of the volume of projections.** The analogue of Proposition 4 for projections, expressing the normalized volume of projections of B_q^n as an L_1-moment of a sum of independent random variables has been established in [3, Proposition 2].

Proposition 5 (Barthe–Naor, [3])**.** *Fix $q \in (1, \infty)$ and $n \in \mathbb{N}$. Let $X_1, X_2, \ldots$ be i.i.d. random variables with density $\gamma_q^{-1}|x|^{\frac{2-q}{q-1}} e^{-|x|^{\frac{q}{q-1}}}$, where $\gamma_q = 2(q-1)\Gamma\big(1 + \frac{1}{q}\big)$. Then, for every unit vector a in $\mathbb{R}^n$ we have*

$$\frac{\mathrm{vol}(\mathrm{Proj}_{a^\perp}\mathsf{B}_q^n)}{\mathrm{vol}(\mathsf{B}_q^{n-1})} = \Gamma\Big(\frac{1}{q}\Big)\mathbb{E}\Big|\sum_{j=1}^n a_j X_j\Big|. \tag{11}$$

When $q = 1$, the identity reduces to the consequence (4) of the Cauchy projection formula.

2.3. **Stability estimates.** As explained in the introduction, a crucial step in the proofs of Theorems 1 and 2 is a reduction to sections and projections with respect to hyperplanes near the extremizer $\left(\frac{e_1+e_2}{\sqrt{2}}\right)^{\perp}$. This will be a consequence of two recent works [7, 6] establishing the stability of the inequalities of Szarek [29] and Ball [1] with respect to the unit normal vector a. For the case of projections, we will use the following robust Szarek inequality proven in [7].

Theorem 6 (De–Diakonikolas–Servedio, [7]). *There exists $\kappa_1 > 0$ such that for every $n \in \mathbb{N}$ and every unit vector a in $\mathbb{R}^n$ with $a_1 \geq \cdots \geq a_n \geq 0$, we have*

$$\mathbb{E}\Big|\sum_{j=1}^{n} a_j \varepsilon_j\Big| \geq \frac{1}{\sqrt{2}} + \kappa_1 \Big|a - \frac{e_1+e_2}{\sqrt{2}}\Big|. \tag{12}$$

We can take $\kappa_1 = 8 \cdot 10^{-5}$ in this inequality.

For the case of sections, we will use the following robust Ball inequality of [6]. We express it in the equivalent negative moment formulation which follows from Proposition 4.

Theorem 7 (Chasapis–Nayar–Tkocz, [6]). *There exists $\kappa_\infty > 0$ such that for every $n \in \mathbb{N}$ and every unit vector a in $\mathbb{R}^n$ with $a_1 \geq \cdots \geq a_n \geq 0$, we have*

$$\mathbb{E}\Big|\sum_{j=1}^{n} a_j \xi_j\Big|^{-1} \leq \sqrt{2} - \kappa_\infty \Big|a - \frac{e_1+e_2}{\sqrt{2}}\Big|. \tag{13}$$

We can take $\kappa_\infty = 6 \cdot 10^{-5}$ in this inequality.

Unfortunately, a direct implementation of the arguments of [7, 6] does not yield explicit values for the constants κ_1 and κ_∞ which are needed for our estimation of p_0 and q_0 in Theorems 1 and 2. In Section 5, we shall present a new short proof of Theorem 6 which is in the spirit of [6] and gives the numerical constant $\kappa_1 = 8 \cdot 10^{-5}$. Moreover, we will explain how to quantify an existential argument used in [6] in order to prove Theorem 7 with $\kappa_\infty = 6 \cdot 10^{-5}$.

2.4. **A technical lemma.** In this section we present the following key lemma, which is crucial for the induction argument sketched in Section 1.3 to work.

Lemma 8. *Let $c \geq 1$ and $p > 4\sqrt{2}c$. If $0 < a_2 \leq a_1$ satisfy $\|(a_1,a_2)\|_p \leq 2^{\frac{1}{p}-\frac{1}{2}}$ and $|a_i - \frac{1}{\sqrt{2}}| \leq \frac{c}{p}$ for $i = 1, 2$, then we have*

$$|a_1 - a_2| \leq 3.65\sqrt{\frac{c}{p-2}}\sqrt{1 - a_1^2 - a_2^2}. \tag{14}$$

To prove it, we need an elementary inequality between p-means with a deficit.

Lemma 9. *Let $\sigma > 0$, $r \geq \max\{\sigma, 2\}$ and $b_1, b_2 \in (0,1]$ with $1 - \frac{\sigma}{r} \leq \frac{b_2}{b_1} \leq 1$. Then, we have*

$$\left(\frac{b_1^r + b_2^r}{2}\right)^{\frac{1}{r}} \geq \frac{b_1+b_2}{2} + (r-1)\frac{1-e^{-\frac{\sigma}{2}}}{4\sigma}|b_1 - b_2|^2. \tag{15}$$

Proof. Denote $c_r \stackrel{\text{def}}{=} (r-1)\frac{1-e^{-\frac{\sigma}{2}}}{4\sigma}$. Dividing both sides by b_1, introducing $\delta \stackrel{\text{def}}{=} 1 - \frac{b_2}{b_1}$, raising the inequality to the power r and using that $b_1 \leq 1$, we see that (15) follows from

$$\frac{1+(1-\delta)^r}{2} \geq \left(1 - \frac{\delta}{2} + c_r\delta^2\right)^r, \qquad \delta \in \left[0, \frac{\sigma}{r}\right].$$

We have equality for $\delta = 0$ and thus it is enough to show that on $[0, \frac{\sigma}{r}]$ the derivatives compare,

$$-\frac{r}{2}(1-\delta)^{r-1} \geq r\left(1 - \frac{\delta}{2} + c_r\delta^2\right)^{r-1}\left(-\frac{1}{2} + 2c_r\delta\right).$$

Multiplying both sides by $\frac{2}{r}$ and rearranging gives an equivalent form

$$1-4c_r\delta \geq \left(\frac{1-\delta}{1-\frac{\delta}{2}+c_r\delta^2}\right)^{r-1},$$

since $1-\frac{\delta}{2}+c_r\delta^2>0$ on $\left[0,\frac{\sigma}{r}\right]$. To prove the last inequality, observe that

$$\left(\frac{1-\delta}{1-\frac{\delta}{2}+c_r\delta^2}\right)^{r-1} \leq \left(\frac{1-\delta}{1-\frac{\delta}{2}}\right)^{r-1} \leq \left(1-\frac{\delta}{2}\right)^{r-1}.$$

It is enough to check the inequality $\left(1-\frac{\delta}{2}\right)^{r-1} \leq 1-4c_r\delta$ only for $\delta\in\{0,\frac{\sigma}{r}\}$, since the left-hand side is convex in δ. For $\delta=\frac{\sigma}{r}$ we have $(1-\frac{\sigma}{2r})^{r-1} \leq e^{-\frac{\sigma}{2}\cdot\frac{r-1}{r}}$, so we would like to prove that

$$e^{-\frac{\sigma}{2}\cdot\frac{r-1}{r}} \leq 1-\frac{r-1}{r}(1-e^{-\frac{\sigma}{2}}).$$

Since $u=\frac{r-1}{r}\in[0,1]$ we want to verify $e^{-\frac{\sigma}{2}u} \leq 1-u(1-e^{-\frac{\sigma}{2}})$, which follows by observing that the left-hand side is a convex function of u and we have equality for $u\in\{0,1\}$. □

Proof of Lemma 8. Since $p>\sqrt{2}c$, we have

$$\frac{a_2}{a_1} \geq \frac{\frac{1}{\sqrt{2}}-\frac{c}{p}}{\frac{1}{\sqrt{2}}+\frac{c}{p}} = \frac{1-\frac{\sqrt{2}c}{p}}{1+\frac{\sqrt{2}c}{p}} \geq \left(1-\frac{\sqrt{2}c}{p}\right)^2 \geq 1-2\sqrt{2}\frac{c}{p},$$

so $\frac{a_2^2}{a_1^2} \geq 1-4\sqrt{2}\frac{c}{p} = 1-\frac{2\sqrt{2}c}{p/2}$. We can apply Lemma 9 with $r=\frac{p}{2}$, $b_i=a_i^2$ and $\sigma=2\sqrt{2}c$ to get

$$\frac{1}{2} \geq \left(\frac{a_1^p+a_2^p}{2}\right)^{\frac{2}{p}} \geq \frac{a_1^2+a_2^2}{2}+\left(\frac{p}{2}-1\right)\frac{1-e^{-\sqrt{2}c}}{8\sqrt{2}c}|a_1^2-a_2^2|^2,$$

where the leftmost inequality is equivalent to $\|(a_1,a_2)\|_p \leq 2^{\frac{1}{p}-\frac{1}{2}}$. By the assumptions, we also have $a_1+a_2 \geq \sqrt{2}-\frac{2c}{p} \geq \sqrt{2}-\frac{1}{2\sqrt{2}}$ and $e^{-c\sqrt{2}}<e^{-\sqrt{2}}$. Therefore, rearranging gives

$$1-a_1^2-a_2^2 \geq \frac{c_0}{c}(p-2)|a_1-a_2|^2, \qquad c_0=\frac{\left(\sqrt{2}-\frac{1}{2\sqrt{2}}\right)^2}{8\sqrt{2}}(1-e^{-\sqrt{2}}).$$

Thus, we conclude that

$$|a_1-a_2| \leq \frac{\sqrt{c}}{\sqrt{c_0(p-2)}}\sqrt{1-a_1^2-a_2^2}, \qquad \frac{1}{\sqrt{c_0}}<3.65,$$

which completes the proof. □

3. Sections

3.1. **Ancillary results.** We begin with a simple L_2-bound quantifying that the distribution of the random magnitudes R_j from (8) is close to the point mass at 1 as p gets large. Explicit computations using the density show that for every $s>-p-1$, the s-th moment of R_1 is

$$\mathbb{E}R_1^s=\frac{\Gamma\left(1+\frac{s+1}{p}\right)}{\Gamma\left(1+\frac{1}{p}\right)}. \tag{16}$$

Lemma 10. *For $p>5$, we have*

$$\mathbb{E}|R_1-1|^2 \leq \frac{2}{\Gamma(1+1/p)}p^{-2}. \tag{17}$$

Proof. By (16), we can write

$$\mathbb{E}|R_1 - 1|^2 = \mathbb{E}R_1^2 - 2\mathbb{E}R_1 + 1 = \frac{\Gamma(1+3/p) - 2\Gamma(1+2/p) + \Gamma(1+1/p)}{\Gamma(1+1/p)}.$$

The function

$$h(x) \stackrel{\text{def}}{=} \Gamma(1+3x) - 2\Gamma(1+2x) + \Gamma(1+x)$$

satisfies $h(0) = h'(0) = 0$, so for every $0 < x < \frac{1}{5}$, by Taylor's expansion with Lagrange's remainder, there exists $0 < \theta < x$ such that

$$h(x) = \frac{1}{2}x^2 h''(\theta) = \frac{1}{2}x^2(9\Gamma''(1+3\theta) - 8\Gamma''(1+2\theta) + \Gamma''(1+\theta)). \tag{18}$$

On the interval $(1, \frac{8}{5})$, Γ'' is decreasing, so $\Gamma''(1+3\theta) < \Gamma''(1+2\theta)$. Since additionally $\Gamma''(s) < 2$ for $s \in (1, \frac{8}{5})$, equation (18) gives $h(x) \le 2x^2$. This applied to $x = \frac{1}{p}$ leads to (17). □

To deal with hyperplanes far from the extremizer, we will crucially rely on the equi-continuity of the section functions at $p = \infty$. For $p \in [1, \infty]$ we introduce the normalized section function,

$$A_{n,p}(a) \stackrel{\text{def}}{=} \frac{\mathrm{vol}(\mathsf{B}_p^n \cap a^\perp)}{\mathrm{vol}(\mathsf{B}_p^{n-1})}, \tag{19}$$

where a is a unit vector in $\mathbb{R}^n$. Additionally, observe that

$$A_{n,\infty}(a) = \frac{\mathrm{vol}(\mathsf{B}_\infty^n \cap a^\perp)}{\mathrm{vol}(\mathsf{B}_\infty^{n-1})} = \mathrm{vol}\big(\mathsf{Q}_n \cap a^\perp\big),$$

where $\mathsf{Q}_n = [-\frac{1}{2}, \frac{1}{2}]^n$ is the unit-volume cube in $\mathbb{R}^n$. Recall that from Proposition 4,

$$A_{n,p}(a) = \Gamma\Big(1 + \frac{1}{p}\Big)\ \mathbb{E}\Big|\sum_{j=1}^n a_j R_j \xi_j\Big|^{-1}.$$

Lemma 11. *Let $p > 5$. For every unit vector a in $\mathbb{R}^n$, we have*

$$\big|A_{n,p}(a) - A_{n,\infty}(a)\big| \le \frac{5}{p}. \tag{20}$$

Proof. First recall that for an arbitrary nonzero vector x in $\mathbb{R}^n$,

$$\mathsf{N}(x) \stackrel{\text{def}}{=} \frac{|x|}{\mathrm{vol}(\mathsf{Q}_n \cap x^\perp)} = \Big(\mathbb{E}\Big|\sum_{j=1}^n x_j \xi_j\Big|^{-1}\Big)^{-1}$$

is a norm by Busemann's theorem [4]. In particular, using $1 \le \mathrm{vol}(\mathsf{Q}_n \cap x^\perp) \le \sqrt{2}$, we get

$$\begin{aligned}\Big|\mathsf{N}(y)^{-1} - \mathsf{N}(x)^{-1}\Big| &= \frac{|\mathsf{N}(x) - \mathsf{N}(y)|}{\mathsf{N}(x)\mathsf{N}(y)} \le \frac{\mathsf{N}(x-y)}{\mathsf{N}(x)\mathsf{N}(y)} \\ &= \frac{|x-y|}{|x||y|}\frac{\mathrm{vol}(\mathsf{Q}_n \cap x^\perp)\mathrm{vol}(\mathsf{Q}_n \cap y^\perp)}{\mathrm{vol}(\mathsf{Q}_n \cap (x-y)^\perp)} \le 2\frac{|x-y|}{|x||y|},\end{aligned}$$

where $x, y \in \mathbb{R}^n \setminus \{0\}$. Evoking (8), we can write

$$\frac{A_{n,p}(a)}{\Gamma(1+1/p)} = \mathbb{E}_R\mathbb{E}_\xi\Big|\sum_{j=1}^n a_j R_j \xi_j\Big|^{-1} = \mathbb{E}_R \mathsf{N}(aR)^{-1},$$

where we use the ad hoc notation aR for the vector $(a_1R_1, \ldots, a_nR_n)$ in $\mathbb{R}^n$. From the previous bound on $1/\mathsf{N}$, we thus obtain

$$\left|\frac{A_{n,p}(a)}{\Gamma(1+1/p)} - A_{n,\infty}(a)\right| = \big|\mathbb{E}\mathsf{N}(aR)^{-1} - \mathsf{N}(a)^{-1}\big| \le 2\mathbb{E}\frac{|a - aR|}{|a|\cdot|aR|} = 2\mathbb{E}\frac{|a-aR|}{|aR|}.$$

By the Cauchy-Schwarz inequality,

$$\mathbb{E}\frac{|a-aR|}{|aR|} \le \sqrt{\mathbb{E}|a-aR|^2}\sqrt{\mathbb{E}|aR|^{-2}} = \sqrt{\mathbb{E}\sum_{j=1}^n a_j^2(R_j-1)^2}\sqrt{\mathbb{E}\Big(\sum_{j=1}^n a_j^2R_j^2\Big)^{-1}}.$$

The first factor in the right-hand side is equal to $\|R_1-1\|_2$. By the convexity of the function $s\mapsto \frac{1}{s}$,

$$\mathbb{E}\Big(\sum_{j=1}^n a_j^2R_j^2\Big)^{-1} \le \sum_{j=1}^n a_j^2\mathbb{E}R_j^{-2} \overset{(16)}{=} \frac{\Gamma\big(1-\frac{1}{p}\big)}{\Gamma\big(1+\frac{1}{p}\big)}.$$

Combining all the above, yields

$$\big|A_{n,p}(a)-\Gamma(1+1/p)A_{n,\infty}(a)\big| \le 2\|R_1-1\|_2\sqrt{\Gamma(1-1/p)\Gamma(1+1/p)}.$$

Using Lemma 10, the right-hand side gets upper-bounded by

$$2\sqrt{\frac{2}{\Gamma(1+1/p)}p^{-2}}\sqrt{\Gamma(1-1/p)\Gamma(1+1/p)} < \frac{2\sqrt{2\Gamma(1/2)}}{p} = \frac{2\sqrt{2}\sqrt[4]{\pi}}{p}$$

using $p>2$. Consequently,

$$\big|A_{n,p}(a)-A_{n,\infty}(a)\big| \le \frac{2\sqrt{2}\sqrt[4]{\pi}}{p} + \big(1-\Gamma(1+1/p)\big)A_{n,\infty}(a) \le \frac{2\sqrt{2}\sqrt[4]{\pi}}{p} + \frac{\sqrt{2}\gamma}{p} < \frac{5}{p},$$

because $1-\Gamma(1+x) < -\Gamma'(1)x = \gamma x$ for $0<x<1$, by concavity. Here, $\gamma = 0.577..$ is the Euler–Mascheroni constant. □

3.2. **Proof of Theorem 1.** Following notation (19), our goal is to prove that for every $p\ge p_0$ and every unit vector a in $\mathbb{R}^n$, we have

$$A_{n,p}(a) \le A_{n,p}\left(\frac{e_1+e_2}{\sqrt{2}}\right), \tag{21}$$

where the right-hand side is explicitly given by

$$A_{n,p}\left(\frac{e_1+e_2}{\sqrt{2}}\right) = \Gamma\Big(1+\frac{1}{p}\Big)\,\mathbb{E}\left|\frac{R_1\xi_1+R_2\xi_2}{\sqrt{2}}\right|^{-1} = A_{2,p}\left(\frac{e_1+e_2}{\sqrt{2}}\right) = \frac{1}{\|(\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}})\|_p} = 2^{\frac{1}{2}-\frac{1}{p}}.$$

Our proof will proceed by induction on n. It is directly checked that the theorem holds when $n=2$, as $A_{2,p}(a)=\|a\|_p^{-1}$ for every unit vector a in $\mathbb{R}^2$. We therefore assume that $n\ge 3$ and $a_1\ge \dots \ge a_n > 0$. Our analysis will differ depending on the distance of a to the extremizer. Let

$$\delta(a) \overset{\text{def}}{=} \left|a-\frac{e_1+e_1}{\sqrt{2}}\right|^2 = 2-\sqrt{2}(a_1+a_2). \tag{22}$$

3.2.1. *The vector a is far from the extremizer.* Suppose that $\sqrt{\delta(a)} \ge \frac{c}{p}$ for some universal constant $c>0$ to be chosen soon. Then, by the equi-continuity proven in Lemma 11 and the stability of Ball's inequality from Theorem 7, we obtain

$$A_{n,p}(a) \le \frac{5}{p} + A_{n,\infty}(a) \le \frac{5}{p} + \sqrt{2} - \kappa_\infty\sqrt{\delta(a)} \le \sqrt{2} - \frac{\kappa_\infty c - 5}{p}.$$

For

$$c \ge \frac{\sqrt{2}\log 2 + 5}{\kappa_\infty} = \frac{\sqrt{2}\log 2+5}{6}10^5 = 0.996..\cdot 10^5,$$

we have

$$\sqrt{2} - \frac{\kappa_\infty c - 5}{p} \le \sqrt{2}\left(1-\frac{\log 2}{p}\right) \le \sqrt{2}e^{-\frac{\log 2}{p}} = 2^{\frac{1}{2}-\frac{1}{p}},$$

which finishes the proof in this case (without using the inductive hypothesis) for $c=10^5$. □

3.2.2. *The vector a is close to the extremizer.* Now, suppose that

$$\sqrt{\delta(a)} < \frac{c}{p},$$

where $c = 10^5$. This in particular implies that (as we already assume that $a_2 \le a_1$),

$$\frac{1}{\sqrt{2}} - \frac{c}{p} \le a_2 \le a_1 \le \frac{1}{\sqrt{2}} + \frac{c}{p}.$$

We shall consider $p > Lc + 2$ for a large constant $L \ge 100$, which we will adjust as we move along. Observe that our goal (21) is equivalent to the inequality

$$\mathbb{E}\Big|\sum_{j=1}^{n} a_j R_j \xi_j\Big|^{-1} \le C_p \tag{23}$$

with

$$C_p = \mathbb{E}\left|\frac{R_1\xi_1 + R_2\xi_2}{\sqrt{2}}\right|^{-1} = \frac{2^{\frac{1}{2}-\frac{1}{p}}}{\Gamma(1+1/p)}. \tag{24}$$

We record for future estimates that when $p > Lc + 2$, we have

$$1.41 < C_p < 1.42, \tag{25}$$

since $2^{10^{-6}} > 2^{1/p}\Gamma(1+1/p) > \Gamma(1+10^{-6})$.

Consider the random vectors $X = a_1R_1\xi_1 + a_2R_2\xi_2$ and $Y = \sum_{j>2} a_jR_j\xi_j$ in $\mathbb{R}^3$. Since X and Y are independent and rotationally invariant, the representation

$$\mathbb{E}\Big|\sum_{j=1}^{n} a_j R_j \xi_j\Big|^{-1} = \mathbb{E}\min\left\{|X|^{-1}, |Y|^{-1}\right\}$$

holds (see, e.g., [6, Lemma 6.6]). By the inductive hypothesis,

$$\mathbb{E}|Y|^{-1} = \frac{1}{\sqrt{1-a_1^2-a_2^2}}\mathbb{E}\left|\frac{\sum_{j>2} a_jR_j\xi_j}{\sqrt{1-a_1^2-a_2^2}}\right|^{-1} \le \frac{C_p}{\sqrt{1-a_1^2-a_2^2}},$$

and hence (by the concavity of the function $t \mapsto \min\{|X|^{-1}, t\}$), we get

$$\mathbb{E}\Big|\sum_{j=1}^{n} a_j R_j \xi_j\Big|^{-1} = \mathbb{E}\min\left\{|X|^{-1}, |Y|^{-1}\right\} \le \mathbb{E}\min\left\{|X|^{-1}, \alpha^{-1}\right\}, \tag{26}$$

where we set

$$\alpha \stackrel{\text{def}}{=} \frac{1}{C_p}\sqrt{1-a_1^2-a_2^2}. \tag{27}$$

Observe that

$$\mathbb{E}\min\left\{|X|^{-1}, \alpha^{-1}\right\} = \mathbb{E}|X|^{-1} - \mathbb{E}\big(|X|^{-1} - \alpha^{-1}\big)_+ \tag{28}$$

and

$$\begin{aligned}\mathbb{E}|X|^{-1} &= \frac{1}{\sqrt{a_1^2+a_2^2}}\mathbb{E}\left|\frac{a_1R_1\xi_1 + a_2R_2\xi_2}{\sqrt{a_1^2+a_2^2}}\right|^{-1} \\ &\overset{(8)}{=} \frac{1}{\sqrt{a_1^2+a_2^2}}\frac{1}{\left\|\frac{(a_1,a_2)}{\sqrt{a_1^2+a_2^2}}\right\|_p \Gamma(1+1/p)} = \frac{1}{\|(a_1,a_2)\|_p\Gamma(1+1/p)}.\end{aligned} \tag{29}$$

In view of the inductive step (26) and the identities (24), (28), (29), the desired inequality (23) is a consequence of the following proposition.

Proposition 12. *Under the assumptions and notation above, for* $p \ge 10^{15}$ *we have*

$$\mathbb{E}\big(|X|^{-1} - \alpha^{-1}\big)_+ \ge C_p\left(\frac{2^{\frac{1}{p}-\frac{1}{2}}}{\|(a_1,a_2)\|_p} - 1\right). \tag{30}$$

Proof. If the right-hand side is nonpositive, we are done. Otherwise,

$$\|(a_1,a_2)\|_p < 2^{\frac{1}{p}-\frac{1}{2}}.$$

Since $\left|a_i - \frac{1}{\sqrt{2}}\right| < \frac{c}{p}$, Lemma 8 gives

$$|a_1 - a_2| \le 3.65\sqrt{\frac{c}{p-2}}\sqrt{1-a_1^2-a_2^2} \overset{(27)}{=} 3.65\sqrt{\frac{c}{p-2}}C_p\alpha \overset{(25)}{\le} \frac{5.25\alpha}{\sqrt{L}}. \tag{31}$$

To simplify, note that $\|(a_1,a_2)\|_p \ge 2^{1/p-1/2}\|(a_1,a_2)\|_2$, so

$$\frac{2^{\frac{1}{p}-\frac{1}{2}}}{\|(a_1,a_2)\|_p} - 1 \le \frac{1}{\|(a_1,a_2)\|_2} - 1 = \frac{1-(a_1^2+a_2^2)}{\sqrt{a_1^2+a_2^2}(1+\sqrt{a_1^2+a_2^2})} < \frac{C_p^2\alpha^2}{1.95},$$

where we used that

$$a_1^2 + a_2^2 \ge 2a_2^2 \ge 2\left(\frac{1}{\sqrt{2}} - \frac{c}{p}\right)^2 > 1 - \frac{2\sqrt{2}c}{p} > 1 - \frac{2\sqrt{2}}{L} > 0.97 \tag{32}$$

and $\sqrt{u}(1+\sqrt{u}) > 1.95$ for $u > 0.97$. Since $\frac{C_p^3}{1.95} < \frac{1.42^3}{1.95} < \frac{3}{2}$, (30) will follow from

$$\mathbb{E}\big(|X|^{-1} - \alpha^{-1}\big)_+ \ge \frac{3}{2}\alpha^2. \tag{33}$$

Consider the event

$$\mathcal{E} \overset{\text{def}}{=} \left\{R_1 \le 1,\ |R_1 - R_2| < \alpha,\ |a_1\xi_1 + a_2\xi_2| < \frac{1}{4}\alpha\right\}.$$

On $\mathcal{E}$, we have

$$\begin{aligned}|X| = |a_1R_1\xi_1 + a_2R_2\xi_2| &\le |a_1R_1\xi_1 + a_2R_1\xi_2| + |a_2R_2\xi_2 - a_2R_1\xi_2| \\ &= R_1|a_1\xi_1 + a_2\xi_2| + a_2|R_2 - R_1| < \frac{1}{4}\alpha + \frac{1}{\sqrt{2}}\alpha < \frac{24}{25}\alpha,\end{aligned}$$

so

$$\mathbb{E}\big(|X|^{-1} - \alpha^{-1}\big)_+ \ge \frac{1}{24\alpha}\mathbb{P}(\mathcal{E}) = \frac{1}{24\alpha}\mathbb{P}\{R_1 \le 1,\ |R_1 - R_2| < \alpha\}\mathbb{P}\left\{|a_1\xi_1 + a_2\xi_2| < \frac{1}{4}\alpha\right\}. \tag{34}$$

For the second probability in (34), observe that the random variable $|a_1\xi_1 + a_2\xi_2|^2$ has the same distribution as $a_1^2 + a_2^2 + 2a_1a_2U$, with U being uniform on $[-1,1]$. Therefore,

$$\mathbb{P}\left\{|a_1\xi_1 + a_2\xi_2| < \frac{1}{4}\alpha\right\} = \mathbb{P}\left\{U < \frac{\alpha^2/16 - a_1^2 - a_2^2}{2a_1a_2}\right\}.$$

Note that the condition

$$-1 < \frac{\alpha^2/16 - a_1^2 - a_2^2}{2a_1a_2} < 1 \tag{35}$$

is equivalent to

$$|a_1 - a_2| < \frac{\alpha}{4} < |a_1 + a_2|.$$

The left inequality holds thanks to (31), provided that $L > (5.25 \cdot 4)^2 = 441$, whereas the right inequality holds since $a_1 + a_2 \ge 2a_2 > \sqrt{2} - \frac{2c}{p} > \sqrt{2} - \frac{2}{L} > 1.2$ which is greater than $\frac{\alpha}{4}$ since

$$\alpha \le \frac{1}{C_p}\sqrt{1-2a_2^2} \overset{(25)}{<} \frac{1}{1.41}\sqrt{1 - 2\left(\frac{1}{\sqrt{2}} - \frac{c}{p}\right)^2} < \frac{1}{1.41}\sqrt{2\sqrt{2}\frac{c}{p}} < \frac{1.2}{\sqrt{L}}. \tag{36}$$

As (35) holds, we have

$$\mathbb{P}\left\{|a_1\xi_1 + a_2\xi_2| < \frac{1}{4}\alpha\right\} = \frac{1}{2}\left(\frac{\alpha^2/16 - a_1^2 - a_2^2}{2a_1a_2} + 1\right) = \frac{\alpha^2/16 - (a_1 - a_2)^2}{4a_1a_2}.$$

Using (31) and the estimate $4a_1a_2 \le 2(a_1^2 + a_2^2) < 2$, we get

$$\mathbb{P}\left\{|a_1\xi_1 + a_2\xi_2| < \frac{1}{4}\alpha\right\} > \frac{1 - 441/L}{32}\alpha^2. \tag{37}$$

For the other probability in (34), it is convenient to place a uniform function of constant mass under the density of R_1, which is doable due to the following technical lemma.

Lemma 13. *Fix $p \in (1,\infty)$ and let $g_p : \mathbb{R}_+ \to \mathbb{R}_+$ be the density of R_1. Then, we have*

$$\forall\, x > 0, \qquad g_p(x) \geq \frac{p}{4}\mathbf{1}_{[1-\frac{1}{2p},1]}(x). \tag{38}$$

Proof. Recall from Proposition 4 that $g_p(x) = p\Gamma(1+1/p)^{-1}x^pe^{-x^p}$ for $x > 0$. Since g_p is log-concave, it suffices to check the inequality at the endpoints $x = 1-\frac{1}{2p}$ and $x = 1$. For the first endpoint, we have

$$g_p\big(1-\tfrac{1}{2p}\big) = \frac{p}{\Gamma(1+1/p)}\big(1-\tfrac{1}{2p}\big)^pe^{-(1-\frac{1}{2p})^p} > \frac{p}{2}e^{-e^{-1/2}} > \frac{p}{4},$$

using that $(1-\frac{1}{2p})^p < e^{-1/2}$ and $(1-\frac{1}{2p})^p > \frac{1}{2}$. Moreover, for the second endpoint,

$$g_p(1) = \frac{p}{e\Gamma(1+1/p)} > \frac{p}{4}.$$

□

Finishing the proof of Proposition 12. We estimate the first probability in (34) using Lemma 13,

$$\begin{aligned}\mathbb{P}\big\{R_1 \leq 1,\ |R_1-R_2| < \alpha\big\} &\geq \iint_{\{x\leq 1,\ |x-y|<\alpha\}} \Big(\frac{p}{4}\Big)^2 \mathbf{1}_{[1-\frac{1}{2p},1]}(x)\mathbf{1}_{[1-\frac{1}{2p},1]}(y)\,\mathrm{d}x\,\mathrm{d}y \\ &= \begin{cases} \frac{1}{64}, & \text{if } \alpha > \frac{1}{2p} \\ \frac{p^2\alpha}{16}\Big(\frac{1}{p}-\alpha\Big), & \text{if } \alpha \leq \frac{1}{2p} \end{cases},\end{aligned} \tag{39}$$

where the equality is an elementary computation. In the case $\alpha \leq \frac{1}{2p}$, we further have $\frac{1}{p}-\alpha \geq \frac{1}{2p}$, so the probability is further bounded from below by $\frac{p\alpha}{32}$, which we will use.

- If $\alpha > \frac{1}{2p}$, inequalities (34), (37) and (39) yield the lower bound

$$\mathbb{E}(|X|^{-1}-\alpha^{-1})_+ \geq \frac{1}{24\alpha}\cdot\frac{1}{64}\cdot\frac{1-441/L}{32}\alpha^2 = \Big(\frac{1-441/L}{2^{14}\cdot 3}\cdot\frac{1}{\alpha}\Big)\alpha^2 \overset{(36)}{>} \Big(\frac{1-441/L}{2^{14}\cdot 3\cdot 1.2}\sqrt{L}\Big)\alpha^2.$$

To get the desired bound (33) by $\frac{3}{2}\alpha^2$, it suffices to take $L = 7.9\cdot 10^9$.

- If $\alpha \leq \frac{1}{2p}$, inequalities (34), (37) and (39) yield the lower bound

$$\mathbb{E}(|X|^{-1}-\alpha^{-1})_+ \geq \frac{1}{24\alpha}\cdot\frac{p\alpha}{32}\cdot\frac{1-441/L}{32}\alpha^2 = \frac{p(1-441/L)}{2^{13}\cdot 3}\alpha^2 > \frac{(L-441)c}{2^{13}\cdot 3}\alpha^2.$$

This is at least $\frac{3}{2}\alpha^2$ for the chosen L, which completes the proof of (33) for $p \geq p_0$, where

$$p_0 = Lc + 2 < 8\cdot 10^9\cdot 10^5 < 10^{15}.$$

□

4. Projections

The proof here parallels the one from Section 3. For the readers' convenience, we include all the details (which are in fact easier in several places).

4.1. **Ancillary results.** We start by quantifying how close the distribution of the X_j from (11) is to that of a Rademacher variable (in the Wasserstein-2 distance). Explicit computations using the density show that for every $s > -\frac{1}{q-1}$, the s-th moment of $|X_1|$ is

$$\mathbb{E}|X_1|^s = \frac{\Gamma\big(1+\frac{(s-1)(q-1)}{q}\big)}{\Gamma\big(\frac{1}{q}\big)}. \tag{40}$$

Lemma 14. *For $1 < q < 2$, we have*

$$\mathbb{E}|X_1 - \mathrm{sgn}(X_1)|^2 \leq 9\Big(1-\frac{1}{q}\Big)^2. \tag{41}$$

Proof. Observe that

$$\mathbb{E}|X_1 - \operatorname{sgn}(X_1)|^2 = \mathbb{E}X_1^2 - 2\mathbb{E}|X_1| + 1 \overset{(40)}{=} \frac{\Gamma(2-1/q) - 2 + \Gamma(1/q)}{\Gamma(1/q)}.$$

Since Γ is decreasing on $(0,1)$, $\Gamma(1/q) \geq \Gamma(1) = 1$. Using Taylor's expansion with Lagrange's remainder, for every $0 < x < 1$ there exists $0 < \theta < x$ such that

$$h(x) \overset{\text{def}}{=} \Gamma(1-x) + \Gamma(1+x) - 2 = \frac{1}{2}x^2\big(\Gamma''(1-\theta) + \Gamma''(1+\theta)\big).$$

Thus for $0 < x < 1/2$, we have

$$h(x) \leq \frac{1}{2}x^2\big(\|\Gamma''\|_{L_\infty(\frac{1}{2},1)} + \|\Gamma''\|_{L_\infty(1,\frac{3}{2})}\big) = \frac{1}{2}x^2\big(\Gamma''(1/2) + \Gamma''(1)\big) < 9x^2$$

since Γ'' decreases on $(\frac{1}{2}, \frac{3}{2})$. Applying this to $x = 1 - \frac{1}{q}$, we indeed obtain

$$\mathbb{E}|X_1 - \operatorname{sgn}(X_1)|^2 \leq h\Big(1 - \frac{1}{q}\Big) \leq 9\Big(1 - \frac{1}{q}\Big)^2.$$ □

From this estimate, we can easily deduce the equi-continuity of the normalized projection functions, which we state directly in probabilistic terms in view of Proposition 5.

Lemma 15. *Let $1 < q < 2$, $X_1, X_2, \ldots$ be i.i.d. random variables from (11) and $\varepsilon_1, \varepsilon_2, \ldots$ be i.i.d. Rademacher random variables. For every unit vector a in $\mathbb{R}^n$, we have*

$$\left|\mathbb{E}\Big|\sum_{j=1}^n a_jX_j\Big| - \mathbb{E}\Big|\sum_{j=1}^n a_j\varepsilon_j\Big|\right| \leq 3\Big(1 - \frac{1}{q}\Big). \tag{42}$$

Proof. Since ε_j has the same distribution as $\operatorname{sgn}(X_j)$, we have

$$\begin{aligned}\left|\mathbb{E}\Big|\sum_{j=1}^n a_jX_j\Big| - \mathbb{E}\Big|\sum_{j=1}^n a_j\varepsilon_j\Big|\right| &= \left|\mathbb{E}\Big|\sum_{j=1}^n a_jX_j\Big| - \mathbb{E}\Big|\sum_{j=1}^n a_j\operatorname{sgn}(X_j)\Big|\right| \\ &\leq \mathbb{E}\Big|\sum_{j=1}^n a_j\big(X_j - \operatorname{sgn}(X_j)\big)\Big| \leq \sqrt{\mathbb{E}\Big|\sum_{j=1}^n a_j\big(X_j - \operatorname{sgn}(X_j)\big)\Big|^2} = \sqrt{\mathbb{E}|X_1 - \operatorname{sgn}(X_1)|^2}\end{aligned}$$

and Lemma 14 finishes the proof. □

4.2. **Proof of Theorem 2.** By virtue of (11), our goal is to show that for every $1 < q < q_0$ and every unit vector a in $\mathbb{R}^n$, we have

$$\mathbb{E}\Big|\sum_{j=1}^n a_jX_j\Big| \geq \mathbb{E}\Big|\frac{X_1 + X_2}{\sqrt{2}}\Big| \overset{\text{def}}{=} c_q. \tag{43}$$

For posterity, we note that thanks to (11), for every vector a in $\mathbb{R}^2$,

$$\mathbb{E}|a_1X_1 + a_2X_2| = |a|\frac{\operatorname{vol}(\operatorname{Proj}_{a^\perp}\mathsf{B}_q^2)}{2\Gamma(1/q)} = \frac{|a|}{\Gamma(1/q)}\sup_{x\in\partial\mathsf{B}_q^2}\left|\left\langle x, \frac{1}{|a|}(-a_2, a_1)\right\rangle\right| = \frac{\|a\|_{\frac{q}{q-1}}}{\Gamma(1/q)}. \tag{44}$$

In particular, we have

$$c_q = \mathbb{E}\Big|\frac{X_1 + X_2}{\sqrt{2}}\Big| = \frac{2^{\frac{1}{2}-\frac{1}{q}}}{\Gamma(1/q)}. \tag{45}$$

In view of the above explicit expression, inequality (43) clearly holds for $n = 2$. We therefore assume that $n \geq 3$, $a_1 \geq \cdots \geq a_n > 0$ and proceed by induction on n. Recall the definition of the deficit parameter used earlier,

$$\delta(a) = \Big|a - \frac{e_1 + e_2}{\sqrt{2}}\Big|^2 = 2 - \sqrt{2}(a_1 + a_2).$$

4.2.1. *The vector a is far from the extremizer.* Here we consider the case $\sqrt{\delta(a)} \geq c\big(1-\frac{1}{q}\big)$ for some constant $c > 0$ to be chosen soon. Using the equi-continuity from Lemma 15 and the robust version of Szarek's inequality from Theorem 6, we obtain

$$\mathbb{E}\Big|\sum_{j=1}^n a_j X_j\Big| \geq \mathbb{E}\Big|\sum_{j=1}^n a_j \varepsilon_j\Big| - 3\Big(1-\frac{1}{q}\Big) \geq \frac{1}{\sqrt{2}} + \kappa_1\sqrt{\delta(a)} - 3\Big(1-\frac{1}{q}\Big) \geq \frac{1}{\sqrt{2}} + (\kappa_1 c - 3)\Big(1-\frac{1}{q}\Big).$$

Note that by convexity, $2^x < 1 + 2(\sqrt{2}-1)x$ for $0 < x < \frac{1}{2}$, which with $x = 1-\frac{1}{q}$ gives

$$c_q = \frac{2^{\frac{1}{2}-\frac{1}{q}}}{\Gamma(1/q)} \leq 2^{\frac{1}{2}-\frac{1}{q}} = \frac{1}{\sqrt{2}}2^{1-\frac{1}{q}} < \frac{1}{\sqrt{2}} + (2-\sqrt{2})\Big(1-\frac{1}{q}\Big).$$

Therefore, if we consider

$$c \geq \frac{5-\sqrt{2}}{\kappa_1} = \frac{5-\sqrt{2}}{8}\cdot 10^5,$$

we get the desired bound (43) (nota bene, without the inductive argument). □

4.2.2. *The vector a is close to the extremizer.* It is left to consider the case when

$$\sqrt{\delta(a)} < c\Big(1-\frac{1}{q}\Big),$$

where $c = \frac{5-\sqrt{2}}{8}\cdot 10^5$. In particular, we also have that

$$\frac{1}{\sqrt{2}} - c\Big(1-\frac{1}{q}\Big) \leq a_2 \leq a_1 \leq \frac{1}{\sqrt{2}} + c\Big(1-\frac{1}{q}\Big). \tag{46}$$

Letting $p = \frac{q}{q-1}$, we shall assume that p is large relative to c, say $p > Lc + 2$ for some large constant $L \geq 100$ to be specified later. In particular, when $\frac{1}{p} = 1 - \frac{1}{q} < 10^{-5}$,

$$0.7 < c_q < 0.71, \tag{47}$$

since $0.7 < \frac{2^{-1/2}}{\Gamma(1-10^{-5})} < \frac{2^{1/2-1/q}}{\Gamma(1/q)} < 2^{-1/2+10^{-5}} < 0.71$.

To run an inductive argument in order to prove (43), we consider the random variables $X = a_1X_1 + a_2X_2$ and $Y = \sum_{j>2} a_j X_j$. By the independence and symmetry of X and Y,

$$\mathbb{E}\Big|\sum_{j=1}^n a_j X_j\Big| = \mathbb{E}|X+Y| = \mathbb{E}\max\{|X|,|Y|\}.$$

Using the inductive hypothesis,

$$\mathbb{E}|Y| = \sqrt{1-a_1^2-a_2^2}\,\mathbb{E}\left|\frac{\sum_{j>2} a_j X_j}{\sqrt{1-a_1^2-a_2^2}}\right| \geq c_q\sqrt{1-a_1^2-a_2^2},$$

hence (by the convexity of the function $t \mapsto \max\{|X|,t\}$), we get

$$\mathbb{E}\Big|\sum_{j=1}^n a_j X_j\Big| = \mathbb{E}\max\{|X|,|Y|\} \geq \mathbb{E}\max\{|X|,\alpha\}, \tag{48}$$

where we set

$$\alpha \stackrel{\text{def}}{=} c_q\sqrt{1-a_1^2-a_2^2}. \tag{49}$$

Observe that

$$\mathbb{E}\max\{|X|,\alpha\} = \mathbb{E}|X| + \mathbb{E}(\alpha - |X|)_+ \tag{50}$$

and, by (44),

$$\mathbb{E}|X| = \frac{\|(a_1,a_2)\|_{\frac{q}{q-1}}}{\Gamma(1/q)} \stackrel{(45)}{=} c_q 2^{\frac{1}{q}-\frac{1}{2}}\|(a_1,a_2)\|_{\frac{q}{q-1}}. \tag{51}$$

In view of the inductive step (48) and the identities (50) and (51), the desired inequality (43) is a consequence of the following proposition.

Proposition 16. *Under the assumptions and notation above, for $1 \le q \le 1 + 10^{-12}$ we have*

$$\mathbb{E}\big(\alpha - |X|\big)_+ \ge c_q\Big(1 - 2^{\frac{1}{q}-\frac{1}{2}}\|(a_1,a_2)\|_{\frac{q}{q-1}}\Big). \tag{52}$$

Proof. If the right-hand side is nonpositive, we are done. Otherwise,

$$\|(a_1,a_2)\|_{\frac{q}{q-1}} < 2^{\frac{1}{2}-\frac{1}{q}}.$$

Letting $p = \frac{q}{q-1}$ and recalling (46), we see that we can apply Lemma 8 to conclude that

$$|a_1 - a_2| \le 3.65\sqrt{\frac{c}{p-2}}\sqrt{1 - a_1^2 - a_2^2} \overset{(49)}{=} \frac{3.65}{c_q}\sqrt{\frac{c}{p-2}}\alpha \overset{(47)}{<} \frac{5.25\alpha}{\sqrt{L}}. \tag{53}$$

To simplify the right-hand side of (52), we write

$$\begin{aligned} c_q\Big(1 - 2^{\frac{1}{q}-\frac{1}{2}}\|(a_1,a_2)\|_{\frac{q}{q-1}}\Big) &\le c_q\big(1 - \|(a_1,a_2)\|_2\big) = c_q\frac{1-(a_1^2+a_2^2)}{1+\sqrt{a_1^2+a_2^2}} \\ &\overset{(49)}{=} \frac{\alpha^2}{c_q(1+\sqrt{a_1^2+a_2^2})} \overset{(47)}{<} \frac{\alpha^2}{0.7(1+\sqrt{0.97})} < \frac{3}{4}\alpha^2, \end{aligned}$$

as we have $a_1^2 + a_2^2 > 0.97$, see (32). Therefore, it suffices to show that

$$\mathbb{E}(\alpha - |X|)_+ \ge \frac{3}{4}\alpha^2. \tag{54}$$

Using that each X_j has the same distribution as $\varepsilon_j|X_j|$, for independent random signs ε_j, we consider the event

$$\mathcal{E} \overset{\text{def}}{=} \left\{|X_1| \le 1,\ \big||X_1| - |X_2|\big| < \alpha,\ |a_1\varepsilon_1 + a_2\varepsilon_2| < \frac{1}{4}\alpha\right\},$$

on which we have

$$|X| = \big|a_1\varepsilon_1|X_1| + a_2\varepsilon_2|X_2|\big| \le |X_1||a_1\varepsilon_1 + a_2\varepsilon_2| + a_2\big||X_2| - |X_1|\big| < \frac{1}{4}\alpha + \frac{1}{\sqrt{2}}\alpha < \frac{24}{25}\alpha$$

and thus we obtain the lower bound

$$\mathbb{E}(\alpha - |X|)_+ \ge \frac{\alpha}{25}\mathbb{P}(\mathcal{E}) = \frac{\alpha}{25}\mathbb{P}\big\{|X_1| \le 1,\ \big||X_1| - |X_2|\big| < \alpha\big\}\mathbb{P}\Big\{|a_1\varepsilon_1 + a_2\varepsilon_2| < \frac{1}{4}\alpha\Big\}. \tag{55}$$

The second probability in (55) is clearly at least $\frac{1}{2}$ provided that

$$|a_1 - a_2| < \frac{\alpha}{4}.$$

This holds assuming $L > (5.25\cdot 4)^2 = 441$, by virtue of (53). For the first probability, analogously to Lemma 13, we will place a constant function under the density f_q of $|X_1|$.

Lemma 17. *Fix $q \in (1, \frac{3}{2})$ and let $f_q : \mathbb{R}_+ \to \mathbb{R}_+$ be the density of $|X_1|$. Then, we have*

$$f_q(x) \ge \frac{1}{4(q-1)}\mathbf{1}_{[1-\frac{q-1}{2},1]}(x), \qquad x > 0. \tag{56}$$

Proof. Recall from Proposition 5,

$$f_q(x) = \frac{1}{(q-1)\Gamma(1+\frac{1}{q})}x^{\frac{2-q}{q-1}}e^{-x^{\frac{q}{q-1}}}, \qquad x > 0.$$

The proof is almost identical to that of Lemma 13. It suffices to check that the inequality holds for $x = 1 - \frac{q-1}{2}$ and $x = 1$. Since $(1 - \frac{q-1}{2})^{\frac{2-q}{q-1}} > 1 - \frac{2-q}{q-1}\frac{q-1}{2} = \frac{q}{2} > \frac{1}{2}$, for $1 < q < \frac{3}{2}$, $(1 - \frac{q-1}{2})^{\frac{q}{q-1}} < e^{-\frac{q}{2}} < e^{-\frac{1}{2}}$ and $\Gamma(1 + \frac{1}{q}) < 1$, we obtain

$$f_q\left(1 - \frac{q-1}{2}\right) > \frac{1}{(q-1)}\frac{1}{2}e^{-\frac{1}{2}} > \frac{1}{4(q-1)}.$$

Moreover,

$$f_q(1) = \frac{1}{(q-1)\Gamma(1+\frac{1}{q})}e^{-1} > \frac{1}{e(q-1)} > \frac{1}{4(q-1)}. \qquad \square$$

Finishing the proof of Proposition 16. As earlier, Lemma 17 gives

$$\begin{aligned}\mathbb{P}\big\{|X_1| \le 1,\ \big||X_1| - |X_2|\big| < \alpha\big\} &\ge \iint_{\{x\le 1,\ |x-y|<\alpha\}} \left(\frac{1}{4(q-1)}\right)^2 \mathbf{1}_{[1-\frac{q-1}{2},1]}(x)\mathbf{1}_{[1-\frac{q-1}{2},1]}(y)\,\mathrm{d}x\,\mathrm{d}y \\ &\ge \begin{cases} \frac{1}{64}, & \alpha > \frac{q-1}{2}, \\ \frac{\alpha}{32(q-1)}, & \alpha \le \frac{q-1}{2}, \end{cases}\end{aligned} \tag{57}$$

where the last inequality follows from (39) with p replaced by $\frac{1}{q-1}$.

• If $\alpha > \frac{q-1}{2}$, inequalities (55) and (57) yield the lower bound

$$\mathbb{E}\big(\alpha - |X|\big)_+ \ge \frac{\alpha}{25}\cdot\frac{1}{64}\cdot\frac{1}{2} = \frac{\alpha}{3200}.$$

Since

$$\alpha \overset{(49)}{\le} c_q\sqrt{1-2a_2^2} \overset{(47)}{<} 0.71\sqrt{1-2\left(\frac{1}{\sqrt{2}}-\frac{c}{p}\right)^2} < 0.71\sqrt{2\sqrt{2}\frac{c}{p}} < \frac{1.2}{\sqrt{L}},$$

we obtain the desired bound (54) provided that $L \ge 8\,294\,400$.

• If $\alpha \le \frac{q-1}{2}$, inequalities (55) and (57) give

$$\mathbb{E}\big(\alpha - |X|\big)_+ \ge \frac{\alpha}{25}\cdot\frac{\alpha}{32(q-1)}\cdot\frac{1}{2} = \frac{\alpha^2}{1600(q-1)}.$$

As we assume $1-\frac{1}{q} \le \frac{1}{cL+2}$, this is at least the desired $\frac{3}{4}\alpha^2$ by a large margin for $L = 8\,294\,400$. The proof is complete for every $1 \le q \le q_0$, where

$$q_0 = \frac{Lc+2}{Lc+1} > 1 + 10^{-12}. \qquad \square$$

5. Stability estimates with explicit constants

The proofs of both Theorems 6 and 7 presented here follow the same strategy taken from [6], which we shall now outline. For a unit vector a in $\mathbb{R}^n$, consider again the deficit

$$\delta(a) = \left|a - \frac{e_1+e_1}{\sqrt{2}}\right|^2 = 2 - \sqrt{2}(a_1+a_2).$$

Let a be a unit vector and without loss of generality assume that $a_1 \ge \cdots \ge a_n \ge 0$. The approach leading to the stability of the inequalities of Szarek and Ball differs depending on whether the vector a is *close to* or *far from* the extremizer $\frac{e_1+e_2}{\sqrt{2}}$, as measured by $\delta(a)$.

Case 1. When a is close to $\frac{e_1+e_2}{\sqrt{2}}$, we quantitatively sharpen the inequalities of Szarek and Ball by reapplying them only to a *portion* of the vector a, thus exhibiting their self-improving feature. The probabilistic formulae are crucial for this part.

When a is far from the extremizer, three things can happen.

Case 2. If the largest magnitude of the coordinates of a is below $\frac{1}{\sqrt{2}}$, the second largest magnitude has to drop *well* below $\frac{1}{\sqrt{2}}$ on the account of $\delta(a)$ being large and the classical Fourier-analytic approach of Haagerup [10] and Ball [1] allows to track the deficit.

Case 3. If the largest magnitude is *barely* above $\frac{1}{\sqrt{2}}$, a Lipschitz property of the section and projection functions allows to reduce this case to the one from Case 2.

Case 4. If the largest magnitude is bounded below away from $\frac{1}{\sqrt{2}}$, an easy convexity/projection argument gives a strict inequality with a margin.

5.1. **Stability of Szarek's inequality.**

We first deal with the sharp Khinchin inequality of [29].

Case 1. We begin with the case that a is near the extremizer.

Lemma 18. *Let* $0 < \delta_0 < \frac{2}{3}$ *and take* $c_0 = \frac{1}{2\sqrt{2}}\left(\sqrt{\frac{1}{5}(4-\delta_0)} - \sqrt{\delta_0}\right) > 0$. *For every unit vector* a *in* $\mathbb{R}^n$ *with* $a_1 \geq \cdots \geq a_n \geq 0$ *satisfying* $\delta(a) \leq \delta_0$, *we have*

$$\mathbb{E}\Big|\sum_{j=1}^n a_j\varepsilon_j\Big| \geq \frac{1}{\sqrt{2}} + c_0\sqrt{\delta(a)}. \tag{58}$$

Proof. We will assume without loss of generality that $n \geq 3$ and $a_1^2 + a_2^2 < 1$ (the remaining cases can be obtained by taking a limit). Let

$$\theta \stackrel{\text{def}}{=} \frac{1}{\sqrt{2}}\sqrt{1 - a_1^2 - a_2^2}.$$

Arguing as in the induction of Section 4 and using Jensen's and Szarek's inequalities, we get

$$\begin{aligned}\mathbb{E}\Big|\sum_{j=1}^n a_j\varepsilon_j\Big| &= \mathbb{E}\max\Big\{|a_1\varepsilon_1 + a_2\varepsilon_2|, \Big|\sum_{j=3}^n a_j\varepsilon_j\Big|\Big\} \geq \mathbb{E}\max\Big\{|a_1\varepsilon_1 + a_2\varepsilon_2|, \mathbb{E}\Big|\sum_{j=3}^n a_j\varepsilon_j\Big|\Big\} \\ &\geq \mathbb{E}\max\big\{|a_1\varepsilon_1 + a_2\varepsilon_2|, \theta\big\} = \frac{1}{2}\max\{a_1 + a_2, \theta\} + \frac{1}{2}\max\{a_1 - a_2, \theta\}.\end{aligned}$$

Denoting by $\delta = \delta(a)$, recall that $\delta = 2 - \sqrt{2}(a_1 + a_2)$, that is,

$$a_1 + a_2 = \frac{2-\delta}{\sqrt{2}}$$

We now claim that $a_1 + a_2 \geq \theta$. This inequality is equivalent to $2 - \delta \geq \sqrt{1 - a_1^2 - a_2^2}$. Since

$$a_1^2 + a_2^2 \geq \frac{1}{2}(a_1 + a_2)^2 = \frac{(2-\delta)^2}{4},$$

it is enough to prove the inequality

$$2 - \delta \geq \sqrt{1 - \frac{(2-\delta)^2}{4}} = \sqrt{\delta - \frac{1}{4}\delta^2}.$$

This is clearly true for $\delta \leq 1$, which holds due to our assumptions.

We therefore want to show the inequality

$$\frac{1}{2}(a_1 + a_2) + \frac{1}{2}\max\{a_1 - a_2, \theta\} \geq \frac{1}{\sqrt{2}} + c_0\sqrt{\delta}.$$

Let us denote $b_1 = a_1 + a_2$ and $b_2 = a_1 - a_2$. Then the desired inequality can be rewritten as

$$\frac{1}{2}b_1 + \frac{1}{2}\max\left\{b_2, \frac{1}{\sqrt{2}}\sqrt{1 - \frac{b_1^2 + b_2^2}{2}}\right\} \geq \frac{1}{\sqrt{2}} + c_0\sqrt{\delta}. \tag{59}$$

• *Assume that* $b_2 \geq \frac{1}{\sqrt{2}}\sqrt{1 - \frac{b_1^2 + b_2^2}{2}}$. This assumption is equivalent with $b_2^2 \geq \frac{2}{5} - \frac{1}{5}b_1^2$ and our goal (59) is to prove that

$$\frac{1}{2}(b_1 + b_2) \geq \frac{1}{\sqrt{2}} + c_0\sqrt{\delta}.$$

Bounding b_2^2 from below by $\frac{2}{5} - \frac{1}{5}b_1^2$ and recalling that $b_1 = \frac{2-\delta}{\sqrt{2}}$, it suffices to prove

$$\frac{2-\delta}{\sqrt{2}} + \sqrt{\frac{2}{5} - \frac{(2-\delta)^2}{10}} \geq \sqrt{2} + 2c_0\sqrt{\delta}.$$

This simplifies to

$$\sqrt{\frac{4-\delta}{5}} \geq 2\sqrt{2}c_0 + \sqrt{\delta},$$

which holds since the function $\eta \mapsto \sqrt{\frac{4-\eta}{5}} - \sqrt{\eta}$ is decreasing on $[0, \delta_0]$.

• *Assume that* $b_2 \leq \frac{1}{\sqrt{2}}\sqrt{1-\frac{b_1^2+b_2^2}{2}}$. This assumption is equivalent with $b_2^2 \leq \frac{2}{5} - \frac{1}{5}b_1^2$ and our goal (59) is to prove that

$$\frac{1}{2}b_1 + \frac{1}{2\sqrt{2}}\sqrt{1-\frac{b_1^2+b_2^2}{2}} \geq \frac{1}{\sqrt{2}} + c_0\sqrt{\delta}.$$

Bounding b_2^2 from above by $\frac{2}{5} - \frac{1}{5}b_1^2$ and recalling that $b_1 = \frac{2-\delta}{\sqrt{2}}$, it suffices to prove

$$\frac{2-\delta}{2\sqrt{2}} + \frac{1}{2\sqrt{2}}\sqrt{\frac{4}{5}\delta - \frac{1}{5}\delta^2} \geq \frac{1}{\sqrt{2}} + c_0\sqrt{\delta}.$$

This simplifies to

$$\sqrt{\frac{4}{5} - \frac{1}{5}\delta} \geq \sqrt{\delta} + 2\sqrt{2}c_0,$$

which holds for the same reason as before. □

Case 2. We assume that a is far from the extremizer and a_1 is at most $\frac{1}{\sqrt{2}}$. A key step in Haagerup's slick Fourier-analytic proof of Szarek's inequality from [10] is the bound

$$\mathbb{E}\Big|\sum_{j=1}^n a_j\varepsilon_j\Big| \geq \sum_{j=1}^n a_j^2 F(a_j^{-2}), \tag{60}$$

for every unit vector a in $\mathbb{R}^n$, where the function $F : (0,\infty) \to \mathbb{R}$ is given by

$$F(s) = \frac{2}{\sqrt{\pi s}} \cdot \frac{\Gamma\left(\frac{s+1}{2}\right)}{\Gamma\left(\frac{s}{2}\right)}, \qquad s > 0.$$

Haagerup showed that the function $F(s)$ is increasing on $(0,\infty)$, which will be crucial for us.

Lemma 19. *Let* $0 < \delta_0 < 2$. *For every unit vector* a *in* $\mathbb{R}^n$ *with* $a_1 \geq \cdots \geq a_n \geq 0$ *satisfying* $\delta(a) \geq \delta_0$ *and* $a_1 \leq \frac{1}{\sqrt{2}}$, *we have*

$$\mathbb{E}\Big|\sum_{j=1}^n a_j\varepsilon_j\Big| \geq \frac{1}{\sqrt{2}} + c_1\sqrt{\delta(a)}, \tag{61}$$

with $c_1 = \frac{1}{2\sqrt{2}}\left(F\left(\frac{8}{(2-\delta_0)^2}\right) - F(2)\right)$.

Proof. We have

$$a_2 \leq \frac{a_1+a_2}{2} = \frac{2-\delta(a)}{2\sqrt{2}} \leq \frac{2-\delta_0}{2\sqrt{2}},$$

which shows that $a_j^{-2} \geq l_0$, for all $j \geq 2$, with $l_0 = \frac{8}{(2-\delta_0)^2} > 2$. Employing (60) and using the monotonicity of F, we therefore have

$$\begin{aligned}\mathbb{E}\Big|\sum_{j=1}^n a_j\varepsilon_j\Big| &\geq a_1^2F(2) + \sum_{j\geq 2} a_j^2 F(l_0) = a_1^2F(2) + (1-a_1^2)F(l_0)\\ &= F(l_0) + a_1^2\big(F(2) - F(l_0)\big) \geq F(l_0) + \frac{1}{2}\big(F(2)-F(l_0)\big)\\ &= \frac{1}{2}\big(F(2)+F(l_0)\big) = \frac{1}{\sqrt{2}} + \frac{1}{2}(F(l_0)-F(2)),\end{aligned}$$

since $F(2) = \frac{1}{\sqrt{2}}$. The conclusion follows since $\delta(a) \leq 2$. □

Case 3. We assume that a is far from the extremizer but a_1 is barely larger than $\frac{1}{\sqrt{2}}$.

Lemma 20. *Let $\gamma_0 \le 1 - \frac{1}{\sqrt{2}}$ and $2\sqrt{\gamma_0} < \delta_0 < 2$. For every unit vector a in $\mathbb{R}^n$ with coordinates $a_1 \ge \dots \ge a_n \ge 0$ satisfying $\frac{1}{\sqrt{2}} \le a_1 \le \frac{1}{\sqrt{2}} + \gamma_0$ and $\delta(a) \ge \delta_0$, we have*

$$\mathbb{E}\Big|\sum_{j=1}^n a_j\varepsilon_j\Big| \ge \frac{1}{\sqrt{2}} + c_2\sqrt{\delta(a)}, \tag{62}$$

with

$$c_2 = \frac{1}{2\sqrt{2}}\left(F\left(\frac{8}{(2+2\sqrt{\gamma_0}-\delta_0)^2}\right) - F(2)\right)\sqrt{\delta_0 - 2\sqrt{\gamma_0}} - \sqrt{2\gamma_0+\gamma_0^2}. \tag{63}$$

Proof. Consider the unit vector

$$b \stackrel{\text{def}}{=} \left(\frac{1}{\sqrt{2}}, \sqrt{a_1^2+a_2^2-\frac{1}{2}}, a_3, \dots, a_n\right).$$

Then by the triangle inequality, we obtain the following Lipschitz property,

$$\begin{aligned}\mathbb{E}\Big|\sum_{j=1}^n a_j\varepsilon_j\Big| &\ge \mathbb{E}\Big|\sum_{j=1}^n b_j\varepsilon_j\Big| - \mathbb{E}\Big|\sum_{j=1}^n (a_j-b_j)\varepsilon_j\Big| \\ &\ge \mathbb{E}\Big|\sum_{j=1}^n b_j\varepsilon_j\Big| - \left(\mathbb{E}\Big|\sum_{j=1}^n (a_j-b_j)\varepsilon_j\Big|^2\right)^{1/2} = \Big|\sum_{j=1}^n b_j\varepsilon_j\Big| - |a-b|.\end{aligned}$$

Note that $b_1 \ge b_2$ and since $b_2 \ge a_2$, also $b_2 \ge b_3 \ge \dots \ge b_n$. Moreover,

$$\sqrt{a_1^2+a_2^2-\frac{1}{2}} - a_2 = \frac{a_1^2-\frac{1}{2}}{\sqrt{a_1^2+a_2^2-\frac{1}{2}}+a_2} \le \sqrt{a_1^2-\frac{1}{2}} \le \sqrt{\sqrt{2}\gamma_0+\gamma_0^2} < \sqrt{2\gamma_0}, \tag{64}$$

thus

$$|a-b|^2 = \left(a_1 - \frac{1}{\sqrt{2}}\right)^2 + \left(\sqrt{a_1^2+a_2^2-\frac{1}{2}} - a_2\right)^2 < \gamma_0^2 + 2\gamma_0.$$

Observe that, since $a_1 \ge \frac{1}{\sqrt{2}}$, we have

$$\begin{aligned}\delta(b) = 2 - \sqrt{2}\left(\frac{1}{\sqrt{2}} + \sqrt{a_1^2+a_2^2-\frac{1}{2}}\right) &= \delta(a) - \sqrt{2}\left(\frac{1}{\sqrt{2}} + \sqrt{a_1^2+a_2^2-\frac{1}{2}} - a_1 - a_2\right) \\ \ge \delta_0 - \sqrt{2}\left(\sqrt{a_1^2+a_2^2-\frac{1}{2}} - a_2\right) &\stackrel{(64)}{>} \delta_0 - 2\sqrt{\gamma_0}.\end{aligned}$$

Thus, applying Lemma 19 to the vector b and using the above estimates, we get

$$\begin{aligned}\mathbb{E}\Big|\sum_{j=1}^n a_j\varepsilon_j\Big| &\ge \frac{1}{\sqrt{2}} + \frac{1}{2\sqrt{2}}\left(F\left(\frac{8}{(2+2\sqrt{\gamma_0}-\delta_0)^2}\right) - F(2)\right)\sqrt{\delta(b)} - \sqrt{2\gamma_0+\gamma_0^2} \\ &\ge \frac{1}{\sqrt{2}} + \frac{1}{2\sqrt{2}}\left(F\left(\frac{8}{(2+2\sqrt{\gamma_0}-\delta_0)^2}\right) - F(2)\right)\sqrt{\delta_0 - 2\sqrt{\gamma_0}} - \sqrt{2\gamma_0+\gamma_0^2}.\end{aligned}$$

Finally, as $a_1 \ge \frac{1}{\sqrt{2}}$, we have $\delta(a) = 2 - \sqrt{2}(a_1+a_2) \le 1 - \sqrt{2}a_2 \le 1$ and the proof is complete. □

Case 4. Finally, there is also a simple bound for the case that a_1 is much larger than $\frac{1}{\sqrt{2}}$.

Lemma 21. *Let $\gamma_0 > 0$. For every unit vector a in $\mathbb{R}^n$ with $a_1 \geq \frac{1}{\sqrt{2}} + \gamma_0$, we have*

$$\mathbb{E}\Big|\sum_{j=1}^{n} a_j\varepsilon_j\Big| \geq \frac{1}{\sqrt{2}} + \gamma_0\sqrt{\delta(a)}. \tag{65}$$

Proof. By Jensen's inequality and the fact that $\delta(a) \leq 1$,

$$\mathbb{E}\Big|\sum_{j=1}^{n} a_j\varepsilon_j\Big| \geq |a_1| \geq \frac{1}{\sqrt{2}} + \gamma_0 \geq \frac{1}{\sqrt{2}} + \gamma_0\sqrt{\delta(a)}.$$ □

Constants. Combining Lemmas 18, 19, 20 and 21 with $\delta_0 = 0.66$ (*almost* the maximal value allowed in Lemma 18) and $\gamma_0 = 8 \cdot 10^{-5}$, we conclude that for all unit vectors a in $\mathbb{R}^n$,

$$\mathbb{E}\Big|\sum_{j=1}^{n} a_j\varepsilon_j\Big| \geq \frac{1}{\sqrt{2}} + \kappa_1\sqrt{\delta(a)}$$

with

$$\kappa_1 \geq \min\{c_0, c_1, c_2, \gamma_0\} > \min\left\{1.7 \cdot 10^{-3}, 1.6 \cdot 10^{-2}, 5.1 \cdot 10^{-4}, 8 \cdot 10^{-5}\right\} = 8 \cdot 10^{-5}.$$

This completes the proof of Theorem 6. □

5.2. **Stability of Ball's inequality.** We now turn to the study of Ball's inequality (1). Throughout this section we denote by $\mathsf{Q}_n = [-\frac{1}{2}, \frac{1}{2}]^n$ the cube of unit volume.

Case 1. We begin with the case that a is near the extremizer.

Lemma 22. *For every $n \geq 2$ and every unit vector a in $\mathbb{R}^n$ with $a_1 \geq \cdots \geq a_n \geq 0$ satisfying $\delta(a) \leq \frac{1}{4}$, we have*

$$\mathrm{vol}\big(\mathsf{Q}_n \cap a^{\perp}\big) \leq \sqrt{2} - c_1\sqrt{\delta(a)}, \tag{66}$$

where $c_1 = 0.12$.

Proof. We can assume that $n \geq 3$ and $a_1^2 + a_2^2 < 1$ (the missing cases follow by taking a limit). Leveraging a self-improving feature of Ball's inequality, the proof of [6, Lemma 6.7] yields

$$\mathrm{vol}\big(\mathsf{Q}_n \cap a^{\perp}\big) \leq \sqrt{2}\max\left\{\left(1 - \delta + \sqrt{\frac{\delta(2-\delta)}{5}}\right)^{-1}, (1-\delta)^{-2}\left(1 - \delta - \frac{\sqrt{\delta(2-\delta)}}{2\sqrt{2}}\right)\right\},$$

where $\delta = \delta(a)$. Denoting the maximum on the right-hand side by $M(\delta)$, we can take

$$c_1 = \inf_{0<\delta<1/4} \sqrt{2}\frac{1 - M(\delta)}{\sqrt{\delta}}.$$

Direct numerical calculations show that $c_1 > 0.12$. □

Cases 2 and 3. We assume that a is far from the extremizer but a_1 is not much larger than $\frac{1}{\sqrt{2}}$.

Lemma 23. *For every $n \geq 2$ and every unit vector a in $\mathbb{R}^n$ with $a_1 \geq \cdots \geq a_n \geq 0$ satisfying $\delta(a) \geq \frac{1}{4}$ and $a_1 \leq \frac{1}{\sqrt{2}} + \gamma_0$, we have*

$$\mathrm{vol}\big(\mathsf{Q}_n \cap a^{\perp}\big) \leq \sqrt{2} - c_2, \tag{67}$$

where $\gamma_0 = 3.2 \cdot 10^{-5}$ and $c_2 = 0.0002$.

Proof. Here the proof relies on Fourier-analytic arguments. For the special function

$$\Psi(s) = \frac{2}{\pi}\sqrt{s}\int_0^\infty \left|\frac{\sin t}{t}\right|^s \mathrm{d}t,$$

Ball showed in [1] that $\Psi(s) < \Psi(2) = \sqrt{2}$, for every $s > 2$. We need a robust version of this estimate. Using the Nazarov–Podkorytov lemma [27], König and Koldobsky [21] proved that

$$\forall\ s \geq \frac{9}{4}, \qquad \Psi(s) \leq \Psi(\infty) = \sqrt{\frac{6}{\pi}} = \sqrt{2}\left(\frac{3}{\pi}\right)^{1/2} \tag{68}$$

(that is, $\theta_0 = \left(\frac{3}{\pi}\right)^{1/2}$ in the notation of [6, Lemma 6.8]). The argument now splits in two cases.

• *Assume that* $a_1 \leq \frac{1}{\sqrt{2}}$. Provided that

$$s(a) \overset{\mathrm{def}}{=} 2\Big(1 - \frac{\delta(a)}{2}\Big)^{-2} \geq \frac{9}{4},$$

which holds as long as $\delta(a) \geq 2\big(1 - \frac{2\sqrt{2}}{3}\big) = 0.11..$, with the aid of (68), the arguments from [6, Lemma 6.8] give the explicit estimate

$$\mathrm{vol}\big(\mathsf{Q}_n \cap a^\perp\big) \leq \left(\frac{3}{\pi}\right)^{1/4}\sqrt{2} = \sqrt{2} - \sqrt{2}(1 - (3/\pi)^{1/4}).$$

Therefore, we can take any

$$c_2 \leq \sqrt{2}(1 - (3/\pi)^{1/4}) = 0.016...$$

• *Assume that* $\frac{1}{\sqrt{2}} < a_1 \leq \frac{1}{\sqrt{2}} + \gamma_0$. Using Busemann's theorem [4], this case is reduced in [6, Lemma 6.8] to the previous range, which yields the bound

$$\mathrm{vol}\big(\mathsf{Q}_n \cap a^\perp\big) \leq \sqrt{2} - \sqrt{2}\min\left\{c_1\sqrt{\tfrac{1}{8} - \sqrt{\gamma_0}}, 1 - (3/\pi)^{1/4}\right\} + 2\sqrt{\gamma_0^2 + 2\gamma_0},$$

where c_1 is the constant from Lemma 22. With the choice of parameters $\gamma_0 = 3.2 \cdot 10^{-5}$ and $c_1 = 0.12$, this estimate yields $\mathrm{vol}(\mathsf{Q}_n \cap a^\perp) \leq \sqrt{2} - 0.00021..$ and thus completes the proof. □

Case 4. Finally, there is also a simple bound for the case that a_1 is much larger than $\frac{1}{\sqrt{2}}$.

Lemma 24. *For every $n \geq 2$ and every unit vector a in $\mathbb{R}^n$ satisfying $a_1 \geq \frac{1}{\sqrt{2}} + \gamma_0$, we have*

$$\mathrm{vol}\big(\mathsf{Q}_n \cap a^\perp\big) \leq \sqrt{2} - \frac{2\gamma_0}{1 + \gamma_0\sqrt{2}}\sqrt{\delta(a)}, \tag{69}$$

where $\gamma_0 = 3.2 \cdot 10^{-5}$.

Proof. By Ball's geometric projection argument (see [1, 27]), we have $\mathrm{vol}(\mathsf{Q}_n \cap a^\perp) \leq \frac{1}{a_1}$. Since $a_1 \geq \frac{1}{\sqrt{2}} + \gamma_0$ and hence $\delta(a) < 1$, we deduce that

$$\mathrm{vol}(\mathsf{Q}_n \cap a^\perp) \leq \frac{1}{\frac{1}{\sqrt{2}} + \gamma_0} = \sqrt{2} - \sqrt{2}\left(1 - \frac{1}{1 + \gamma_0\sqrt{2}}\right) \leq \sqrt{2} - \frac{2\gamma_0}{1 + \gamma_0\sqrt{2}}\sqrt{\delta(a)}. \qquad \square$$

Constants. Combining Lemmas 22, 23 and 24, and using that always $\delta(a) < 2$, we conclude that for all unit vectors a in $\mathbb{R}^n$, we have the inequality

$$\mathrm{vol}\big(\mathsf{Q}_n \cap a^\perp\big) \leq \sqrt{2} - \kappa_\infty\sqrt{\delta(a)}$$

with

$$\kappa_\infty \geq \min\left\{c_1, \frac{c_2}{\sqrt{2}}, \frac{2\gamma_0}{1 + \gamma_0\sqrt{2}}\right\} > 6 \cdot 10^{-5}.$$

This completes the proof of Theorem 7. □

Remark 25. We would like to stress that the arguments of this paper have not been optimized to give the best possible constants p_0 and q_0 in Theorems 1 and 2. We instead chose to be fairly generous in various parts of the proof for the sake of clarity of the exposition.

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