A randomly weighted minimum arborescence with a random cost constraint

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Abstract

We study the minimum spanning arborescence problem on the complete digraph $\vec{K}_n$ where an edge e has a weight W_e and a cost C_e, each of which is an independent uniform $[0,1]$ random variable. There is also a constraint that the spanning arborescence T must satisfy $C(T) \leq c_0$. We establish the asymptotic value of the optimum weight via the consideration of a dual problem. The proof is via the analysis of a polynomial time algorithm.

2010 Mathematics Subject Classification. 05C80, 90C27.

Key words. Random Minimum Spanning Arborescence, Cost Constraint.

1 Introduction

We consider the minimum spanning arborescence problem in the context of the complete digraph $\vec{K}_n$ where each edge has an independent uniform $[0,1]$ weight W_e and an independent uniform $[0,1]$ cost C_e. Let $\mathcal{A}$ denote the set of spanning arborescences of $\vec{K}_n$. An arborescence is a rooted tree in which every edge is directed away from the root. The weight of a spanning arborescence A is given by $W(A) = \sum_{e\in A} W_e$ and its cost $C(A)$ is given by $C(A) = \sum_{e\in A} C_e$. The problem we study is

$$\text{Minimise } W(A) \text{ subject to } A \in \mathcal{A},\, C(A) \leq c_0, \tag{1}$$

where c_0 may depend on n.

Without the constraint $C(A) \leq c_0$, we have a weighted matroid intersection problem and as such it is solvable in polynomial time, see for example Lawler [6]. Furthermore Edmonds [2] gave a particularly elegant algorithm for solving this problem. With the constraint $C(A) \leq c_0$, the problem becomes NP-hard, since the knapsack probelm can be easily reduced to it. On the other hand, equation (1) defines a natural problem that has been considered in the literature,

*Research supported in part by NSF grant DMS1661063

in the worst-case rather than the average case. See for example Guignard and Rosenwein [5] and Aggarwal, Aneja and Nair [1] and Goemans and Ravi [4] (for an undirected version). This paper is a follow up to the analysis of the cost constrained minimum weight spanning tree problem considered in [3].

The addition of a cost contraint makes the problem NP-hard and reflects the fact that in many practical situations there may be more than one objective for an optimization problem. Here the goal is to lower weight and cost.

Theorem 1. *Let D_n be the complete digraph $\vec{K}_n$ on n vertices with each edge e having assigned a random weight W_e and a random cost C_e, where $\{W_e, C_e\}$ is a family of i.i.d. random variables uniform on $[0,1]$. Given $c_0 > 0$, let W^*_{arb} be the optimum value for the problem (1). The following hold w.h.p.*

Case 1: *If $c_0 \in \sqrt{\frac{\pi}{8}}[\sqrt{\log n}, \frac{n}{(\log n)^2}]$, then*

$$W^*_{arb} \approx \frac{\pi n}{8c_0}.$$

Case 2: *Suppose now that $c_0 = \alpha n$, where $\alpha = O(1)$ is a positive constant.*

(i) *If $\alpha > 1/2$ then*

$$W^*_{arb} \approx 1.$$

(ii) *If $\alpha < 1/2$ then*

$$W^*_{arb} \approx f(\beta^*) - \alpha\beta^*$$

where β^* is the unique positive solution to $f'(\beta) = \alpha$ and where

$$f(\beta) = \beta^{1/2} \int_{t=0}^{\beta^{1/2}} e^{-t^2/2} dt + e^{-\beta/2}, \qquad \beta > 0.$$

Case 3: *Suppose now that $c_0 = \alpha$, where $\alpha = O(1)$ is a positive constant.*

(i) *If $\alpha < 1$ then there is no solution to (1).*

(ii) *If $\alpha > 1$ then*

$$W^*_{arb} \approx (g(\beta) - \alpha\beta)n$$

where β is the unique positive solution to $g'(\beta) = \alpha$ and where

$$\begin{aligned} g(\beta) &= \beta^{1/2} \int_{t=0}^{\beta^{-1/2}} e^{-t^2/2} dt + \beta e^{-1/2\beta} \\ &= \beta f(1/\beta), \qquad\qquad \beta > 0. \end{aligned}$$

We note that Lemma 2 of Section 2.1 shows that the claims in Case 2 are reasonable and Lemma 3 shows that the claims in Case 3 are reasonable (that is, the stated equations possses unique solutions).

2 Auxiliary results

2.1 Properties of the functions f and g

Lemma 2. *$f(0) = 1, f(\infty) = \infty$, $f'(0) = 1/2, f'(\infty) = 0$ and f' is strictly monotone decreasing. These imply that $f' > 0$, f is concave increasing and for every $0 < \alpha < \frac{1}{2}$, there is a unique $\beta > 0$ such that $f'(\beta) = \alpha$.*

Proof. This follows by inspection of f and

$$f'(\beta) = \frac{1}{2\beta^{1/2}} \int_{t=0}^{\beta^{1/2}} e^{-t^2/2} \mathrm{d}t.$$
$$f''(\beta) = \frac{1}{4\beta^{3/2}} \int_{t=0}^{\beta^{1/2}} \left(e^{-\beta/2} - e^{-t^2/2}\right) \mathrm{d}t < 0.$$

□

Lemma 3. *$g'(0) = \infty, g'(\infty) = 1$ and g' is strictly monotone decreasing. This implies that g is concave and for every $\alpha > 1$, there is a unique $\beta > 0$ such that $g'(\beta) = \alpha$.*

Proof. We have $g(\beta) = \beta f(1/\beta)$ and

$$g'(\beta) = f(1/\beta) - \frac{1}{\beta} f'(1/\beta) = \frac{1}{2\beta^{1/2}} \int_{t=0}^{\beta^{-1/2}} e^{-t^2/2} \mathrm{d}t + e^{-1/2\beta}.$$
$$g''(\beta) = \frac{1}{\beta^3} f''(1/\beta) < 0.$$

By inspection, $g'(0) = \infty$ and $g'(\infty) = 1$. □

2.2 Expectation

Our strategy will be to prove results about mappings $f : [n] \to [n]$, where $f(i) \neq i, i \in [n]$. Given f, we have a digraph D_f with vertex set $[n]$ and edge set $A_f = \{(i, f(i)) : i \in [n]\}$. Most of the analysis concerns the problem
Minimum Weight Constrained Mapping (MWCM):

$$\text{Minimise } W_{map}(f) = \sum_{i \in [n]} W_{(i,f(i))} \text{ subject to } C(f) = \sum_{i \in [n]} C_{(i,f(i))} \leq c_0.$$

Let f^* solve MWCM. We will argue that w.h.p. D_{f^*} is close to being an arboresence and that a small change will result in a near optimum arborescence that will verify the claims of Theorem 1. The following lemma begins our analysis of optimal mappings. We have expressed the following calculations with n replacing $n-1$, but this does not affect the final results.

Lemma 4. *Let $X_1, X_2, \ldots$ and $Y_1, Y_2, \ldots$ be i.i.d. random variables uniform on $[0, 1]$. Then*

E1: *For* $\lambda \le \frac{1}{n\log n}$*, we have*

$$\mathbb{E}\min_{i\le n}\{X_i+\lambda Y_i\} = (1+o(1))\frac{1}{n}. \tag{2}$$

E2: *For* $\frac{1}{n\log n} \le \lambda \le \frac{\log n}{n}$*, we have*

$$\mathbb{E}\min_{i\le n}\{X_i+\lambda Y_i\} = (1+o(1))\frac{1}{n}\left(\sqrt{\lambda n}\int_0^{\sqrt{\lambda n}} e^{-\frac{t^2}{2}}\mathrm{d}t + e^{-\lambda n/2}\right). \tag{3}$$

E3: *For* $\frac{\log n}{n} \le \lambda \le \frac{n}{\log n}$*, we have*

$$\mathbb{E}\min_{i\le n}\{X_i+\lambda Y_i\} = (1+o(1))\sqrt{\frac{\pi}{2}}\sqrt{\frac{\lambda}{n}}. \tag{4}$$

E4: *For* $\frac{n}{\log n} \le \lambda \le n\log n$*, we have*

$$\mathbb{E}\min_{i\le n}\{X_i+\lambda Y_i\} = (1+o(1))\frac{\lambda}{n}\left(\sqrt{\frac{n}{\lambda}}\int_0^{\sqrt{\frac{n}{\lambda}}} e^{-\frac{t^2}{2}}\mathrm{d}t + e^{-\frac{1}{2}\frac{n}{\lambda}}\right). \tag{5}$$

E5: *For* $\lambda \ge n\log n$*, we have*

$$\mathbb{E}\min_{i\le n}\{X_i+\lambda Y_i\} = (1+o(1))\frac{\lambda}{n}. \tag{6}$$

Proof. Thanks to independence

$$\begin{aligned}\mathbb{E}\min_{i\le n}\{X_i+\lambda Y_i\} &= \int_0^\infty \mathbb{P}\left(\min_{i\le n}\{X_i+\lambda Y_i\} > t\right)\mathrm{d}t \\ &= \int_0^\infty \Big[\mathbb{P}\left(X_1+\lambda Y_1 > t\right)\Big]^n \mathrm{d}t.\end{aligned}$$

Case 1. $\lambda \ge 1$.
It follows from an elementary computation that (for details see e.g. the appendix in [3])

$$\mathbb{P}\left(X_1+\lambda Y_1 > t\right) = \begin{cases} 1-\frac{t^2}{2\lambda}, & 0<t<1, \\ 1+\frac{1}{2\lambda}-\frac{t}{\lambda}, & 1\le t<\lambda, \\ \frac{(1+\lambda-t)^2}{2\lambda}, & \lambda\le t<1+\lambda, \\ 0, & t\ge 1+\lambda. \end{cases}$$

Thus,

$$\begin{aligned}\mathbb{E}\min_{i\le n}\{X_i+\lambda Y_i\} &= \int_0^1 \left(1-\frac{t^2}{2\lambda}\right)^n \mathrm{d}t \\ &\quad + \int_1^\lambda \left(1+\frac{1}{2\lambda}-\frac{t}{\lambda}\right)^n \mathrm{d}t \\ &\quad + \int_\lambda^{1+\lambda} \left(\frac{(1+\lambda-t)^2}{2\lambda}\right)^n \mathrm{d}t \\ &= \int_0^1 \left(1-\frac{t^2}{2\lambda}\right)^n \mathrm{d}t \qquad (7) \\ &\quad + \frac{\lambda}{n+1}\left[\left(1-\frac{1}{2\lambda}\right)^{n+1} - \left(\frac{1}{2\lambda}\right)^{n+1}\right] + \frac{1}{2n+1}\left(\frac{1}{2\lambda}\right)^{2n}.\end{aligned}$$

Case 1.1. $1 \le \lambda \le \frac{n}{\log n}$
A change of variables gives

$$\int_0^1 \left(1 - \frac{t^2}{2\lambda}\right)^n \mathrm{d}t = \sqrt{\lambda} \int_0^{\frac{1}{\sqrt{\lambda}}} \left(1 - \frac{t^2}{2}\right)^n \mathrm{d}t. \tag{8}$$

We have $\sqrt{\log n/n} < \frac{1}{\sqrt{\lambda}} < 1$ and

$$\int_{\sqrt{\log n/n}}^{\infty} \left(1 - \frac{t^2}{2}\right)^n \mathrm{d}t \le \int_{\sqrt{\log n/n}}^{\infty} e^{-\frac{nt^2}{2}} \mathrm{d}t = \frac{1}{\sqrt{n}} \int_{\sqrt{\log n}}^{\infty} e^{-\frac{t^2}{2}} \mathrm{d}t = o(n^{-1/2}).$$

Therefore

$$\sqrt{\lambda} \int_0^{\frac{1}{\sqrt{\lambda}}} \left(1 - \frac{t^2}{2}\right)^n \mathrm{d}t = \sqrt{\lambda} \int_0^{\sqrt{\log n/n}} \left(1 - \frac{t^2}{2}\right)^n \mathrm{d}t + \sqrt{\lambda} o(n^{-1/2}).$$

Using $1 + x = e^{x + O(x^2)}$ as $x \to 0$, we get

$$\begin{aligned}
\int_0^{\sqrt{\log n/n}} \left(1 - \frac{t^2}{2}\right)^n \mathrm{d}t &= \int_0^{\sqrt{\log n/n}} e^{-\frac{nt^2}{2} + O(nt^4)} \mathrm{d}t \\
&= (1 + o(1)) \int_0^{\sqrt{\log n/n}} e^{-\frac{nt^2}{2}} \mathrm{d}t \\
&= (1 + o(1)) \frac{1}{\sqrt{n}} \int_0^{\sqrt{\log n}} e^{-\frac{t^2}{2}} \mathrm{d}t \\
&= (1 + o(1)) \frac{1}{\sqrt{n}} \int_0^{\infty} e^{-\frac{t^2}{2}} \mathrm{d}t + o(n^{-1/2}) \\
&= (1 + o(1)) \frac{1}{\sqrt{n}} \sqrt{\frac{\pi}{2}} + o(n^{-1/2}).
\end{aligned}$$

Putting these together back into (8) yields

$$\int_0^1 \left(1 - \frac{t^2}{2\lambda}\right)^n \mathrm{d}t = (1 + o(1)) \sqrt{\frac{\pi}{2}} \sqrt{\frac{\lambda}{n}} + \sqrt{\lambda} o(n^{-1/2}) = (1 + o(1)) \sqrt{\frac{\pi}{2}} \sqrt{\frac{\lambda}{n}}.$$

Since

$$\begin{aligned}
\frac{\lambda}{n+1} \left[\left(1 - \frac{1}{2\lambda}\right)^{n+1} - \left(\frac{1}{2\lambda}\right)^{n+1} \right] + \frac{1}{2n+1} \left(\frac{1}{2\lambda}\right)^{2n} &= O\left(\frac{\lambda}{n}\right) \\
&= \sqrt{\frac{\lambda}{n}} O\left(\sqrt{\frac{1}{\log n}}\right),
\end{aligned}$$

from (7) we can finally obtain (4).

Case 1.2. $\frac{n}{\log n} \le \lambda \le n \log n$
Since for $t \le \frac{1}{\sqrt{\lambda}}$, $(1 - \frac{t^2}{2})^n = e^{-\frac{nt^2}{2}} e^{O(nt^4)} = e^{-\frac{nt^2}{2}} e^{O(\frac{\log^2 n}{n})}$, directly from (8), we get

$$\int_0^1 \left(1 - \frac{t^2}{2\lambda}\right)^n \mathrm{d}t = (1 + o(1)) \sqrt{\lambda} \int_0^{\frac{1}{\sqrt{\lambda}}} e^{-\frac{nt^2}{2}} \mathrm{d}t = (1 + o(1)) \sqrt{\frac{\lambda}{n}} \int_0^{\sqrt{\frac{n}{\lambda}}} e^{-\frac{t^2}{2}} \mathrm{d}t.$$

Moreover,

$$\begin{aligned}
&\frac{\lambda}{n+1}\left[\left(1-\frac{1}{2\lambda}\right)^{n+1}-\left(\frac{1}{2\lambda}\right)^{n+1}\right]+\frac{1}{2n+1}\left(\frac{1}{2\lambda}\right)^{2n}\\
&\quad=(1+o(1))\frac{\lambda}{n}e^{-\frac{n}{2\lambda}+O\left(\frac{n}{\lambda^2}\right)}+O\left(\left(\frac{\log n}{n}\right)^n\right)\\
&\quad=\frac{\lambda}{n}e^{-\frac{n}{2\lambda}}\left(1+o(1)+\frac{n}{\lambda}e^{\frac{n}{2\lambda}}O\left(\left(\frac{\log n}{n}\right)^n\right)\right)\\
&\quad=\frac{\lambda}{n}e^{-\frac{n}{2\lambda}}\left(1+o(1)\right).
\end{aligned}$$

Plugging these back in (7) yields (5).

Case 1.3. $\lambda \geq n\log n$
Plainly,

$$\int_0^1\left(1-\frac{t^2}{2\lambda}\right)^n \mathrm{d}t = O(1) = \frac{\lambda}{n}o(1).$$

Since $\left(1-\frac{1}{2\lambda}\right)^{n+1} = e^{O\left(\frac{n}{\lambda}\right)} = 1+o(1)$, we have

$$\frac{\lambda}{n+1}\left[\left(1-\frac{1}{2\lambda}\right)^{n+1}-\left(\frac{1}{2\lambda}\right)^{n+1}\right]+\frac{1}{2n+1}\left(\frac{1}{2\lambda}\right)^{2n}=\frac{\lambda}{n}\left(1+o(1)\right).$$

Putting these in (7) gives (6).

Case 2. $\lambda \leq 1$
We write

$$\mathbb{E}\min_{i\leq n}\{X_i+\lambda Y_i\} = \lambda\mathbb{E}\min_{i\leq n}\{X_i+\lambda^{-1}Y_i\}$$

and then apply (4), (5) and (6) to λ^{-1}, multiply the answers by λ to get (2), (3) and the missing range $\frac{\log n}{n} \leq \lambda \leq 1$ of (4). □

Corollary 5. *Under the assumptions of Lemma 4, we have*

$$n\mathbb{E}\min_{i\leq n}\{X_i+\lambda Y_i\} = \Omega(\max\{1,\sqrt{\lambda n}\}).$$

Proof. This follows directly from (2) - (6) and the fact that $f(\beta) \geq 1$ (Lemma 2) as well as the lower bound

$$\begin{aligned}
f(\beta) \geq \max\{\sqrt{\beta}\int_0^{\sqrt{\beta}} e^{-t^2/2}\mathrm{d}t, e^{-\beta/2}\} &\geq \max\left\{\sqrt{\beta}\int_0^{\sqrt{\beta}} e^{-t^2/2}\mathrm{d}t, \sqrt{\beta}\mathbf{1}_{\{\beta\leq\frac{1}{2}\}}\right\}\\
&\geq \frac{1}{2}\sqrt{\beta}.
\end{aligned}$$

□

2.3 Concentration

Again n replaces $n-1$ in the calculations.

Lemma 6. *Let $W_{(i,j)}$ and $C_{(i,j)}$, $i,j \le n$, be i.i.d. random variables uniform on $[0,1]$. Let $\lambda \in [0, n\log n]$. For $X_i = \min_j\{W_{(i,j)} + \lambda C_{(i,j)}\}$, $S = \sum_{i\le n} X_i$ and $\varepsilon = \Omega(n^{-1/5})$, we have*

$$\mathbb{P}\left(|S - \mathbb{E}S| > \varepsilon \mathbb{E}S\right) = O(n^{-99}). \tag{9}$$

Moreover,

$$\mathbb{P}\left(\exists i:\ X_i > 10(1+\lambda)\sqrt{\log n/n}\right) \le n^{-99}. \tag{10}$$

Proof. Let $M = 10(1+\lambda)\sqrt{\log n/n}$ and B be the event that for some i, $X_i \ge M$. We have,

$$\mathbb{P}\left(|S - \mathbb{E}S| > \varepsilon \mathbb{E}S\right) \le \mathbb{P}\left(B\right) + \mathbb{P}\left(\left(|S - \mathbb{E}S| > \varepsilon \mathbb{E}S\right) \wedge B^c\right). \tag{11}$$

First we bound $\mathbb{P}\left(B\right)$. By the union bound and independence,

$$\mathbb{P}\left(B\right) \le n\mathbb{P}\left(X_1 \ge M\right) = n\left[\mathbb{P}\left(W_{(1,1)} + \lambda C_{(1,1)} \ge M\right)\right]^n.$$

We use $W_{(1,1)} + \lambda C_{(1,1)} \le (1+\lambda)\max\{W_{(1,1)}, C_{(1,1)}\}$ and note that since these variables are uniform, we have $\mathbb{P}\left(\max\{W_{(1,1)}, C_{(1,1)}\} \ge u\right) = 1 - u^2$ for $u < 1$. We thus get

$$\mathbb{P}\left(B\right) \le n\left[1 - 100\frac{\log n}{n}\right]^n \le ne^{-100\log n} = n^{-99},$$

which establishes (10).

The second term in (11) can be bounded using Chernoff's inequality because on B^c, $X_i = X_i \mathbf{1}_{X_i \le M}$, that is S can be treated as a sum of n independent random variables $\tilde{X}_i = X_i \mathbf{1}_{X_i \le M}$ with $\tilde{X}_i \in [0, M]$. Clearly $\tilde{X}_i \le X_i$ and $\tilde{S} = \sum \tilde{X}_i \le S$, so

$$\mathbb{P}\left(\left(|S - \mathbb{E}S| > \varepsilon \mathbb{E}S\right) \wedge B^c\right) = \mathbb{P}\left(\left(|\tilde{S} - \mathbb{E}S| > \varepsilon \mathbb{E}S\right) \wedge B^c\right) \le \mathbb{P}\left(|\tilde{S} - \mathbb{E}S| > \varepsilon \mathbb{E}S\right).$$

By the Chernoff bound

$$\mathbb{P}\left(|\tilde{S} - \mathbb{E}\tilde{S}| > \varepsilon \mathbb{E}\tilde{S}\right) \le 2\exp\left\{-\frac{\varepsilon^2 \mathbb{E}\tilde{S}}{3M}\right\}.$$

Note that

$$|\tilde{S} - \mathbb{E}S| \le |\tilde{S} - \mathbb{E}\tilde{S}| + |\mathbb{E}S - \mathbb{E}\tilde{S}|.$$

and

$$\begin{aligned} |\mathbb{E}S - \mathbb{E}\tilde{S}| = \left|\mathbb{E}\sum X_i \mathbf{1}_{X_i > M}\right| \le (1+\lambda)\mathbb{E}\sum \mathbf{1}_{X_i > M} &\le (1+\lambda)n\mathbb{P}\left(X_1 > M\right) \\ &= O(n^{-90}), \end{aligned}$$

thanks to (10). Moreover, by Corollary 5,

$$\mathbb{E}S = \Omega(\max\{1, \sqrt{\lambda n}\}),$$

thus

$$|\mathbb{E}S - \mathbb{E}\tilde{S}| \leq \frac{1}{2}\varepsilon\mathbb{E}S$$

and we get

$$\mathbb{P}\left(|\tilde{S} - \mathbb{E}S| > \varepsilon\mathbb{E}S\right) \leq \mathbb{P}\left(|\tilde{S} - \mathbb{E}\tilde{S}| > \frac{1}{2}\varepsilon\mathbb{E}S\right) \leq \mathbb{P}\left(|\tilde{S} - \mathbb{E}\tilde{S}| > \frac{1}{2}\varepsilon\mathbb{E}\tilde{S}\right)$$
$$\leq 2\exp\left\{-\frac{\varepsilon^2\mathbb{E}\tilde{S}}{12M}\right\}.$$

Finally, observe that

$$\frac{\mathbb{E}\tilde{S}}{M} \geq \frac{\mathbb{E}S}{2M} = \frac{\Omega(\max\left\{1, \sqrt{\lambda n}\right\})}{20(1+\lambda)\sqrt{\log n}}\sqrt{n}$$

and for $\lambda \leq n\log n$, we have $\frac{\max\{1,\sqrt{\lambda n}\}}{1+\lambda} \geq \frac{1}{2}\sqrt{\frac{1}{\log n}}$. Consequently,

$$\frac{\varepsilon^2\mathbb{E}\tilde{S}}{12M} = \Omega\left(\frac{\varepsilon^2\sqrt{n}}{\log n}\right) = \Omega(n^{1/10}),$$

so

$$\mathbb{P}\left(|S - \mathbb{E}S| > \varepsilon\mathbb{E}S, B^c\right) = O(e^{-n^{1/10}}).$$

In view of (11), this combined with (10) finishes the proof of (9). □

Corollary 7. *Let M_n denote the minimum weight of a mapping with weights $W_e + \lambda C_e, e \in E(\vec{K}_n)$. Then with probability $1 - O(n^{-90})$,*

$$M_n \approx \begin{cases} (\pi\lambda n/2)^{1/2} & \textbf{E3}. \\ f(\lambda n) & \textbf{E2}. \\ ng(\lambda/n) & \textbf{E4}. \end{cases}$$

$$W_{\max} \leq \begin{cases} O\left((1+\lambda)\sqrt{\log n/n}\right) & \textbf{E3}. \\ O\left(\sqrt{\log n/n}\right) & \textbf{E2}. \\ 1 & \textbf{E4}. \end{cases} \tag{12}$$

$$C_{\max} \leq \begin{cases} O\left(\frac{1}{\lambda}+1\right)\sqrt{\log n/n} & \textbf{E3}. \\ 1 & \textbf{E2}. \\ O(\log n/n) & \textbf{E4}. \end{cases} \tag{13}$$

Proof. The claim about M_n follows directly from Lemma 4 and Lemma 6. For Cases 1 and 2 the claim about $W_{\max}$ follows from (10). For Case 1 the claim about $C_{\max}$ follows from (10). For Case 3, we let $p = K\log n/n$ and argue that w.h.p. for each $v \in [n]$, there exists $w \neq v$ such that $C_{(v,w)} \leq p$ (the probability of the contrary is at most $n(1-p)^{n-1} = o(1)$). If $C_{\max} = C_{(v_1,w_1)} > 2p$ then replacing (v_1, w_1) by (v_1, w_2) where $C_{(v_1,w_2)} \leq p$ we reduce the value $W(F) + \lambda C(F)$ of the supposed mapping F, by at least $\lambda p - 1 \geq \frac{n}{\log n}K\frac{\log n}{n} - 1 > 0$, contradicting the optimality of F. □

2.4 Properties of optimal dual solutions

Let

$$I = \left\{(i,j) \in [n]^2 : i \neq j \text{ and } W_{i,j}, C_{i,j} \text{ are bounded by (12), (13) respectively}\right\}.$$

For $i \in [n]$ we let $J_i = \{j : (i,j) \in I\}$.

We can express the problem MWCM as the following integer program:
IP_{map}

$$\text{Minimize } \sum_{(i,j)\in I} W_{i,j} x_{i,j} \text{ subject to}$$

$$\sum_{j \in J_i} x_{i,j} = 1, i \in [n] \tag{14}$$

$$\sum_{(i,j)\in[I]} C_{i,j} x_{i,j} \leq c_0 \tag{15}$$

$$x_{i,j} = 0 \text{ or } 1, \quad \text{for all } i \neq j. \tag{16}$$

We obtain the relaxation LP_{map} by replacing (16) by

$$0 \leq x_{i,j} \leq 1 \text{ for all } (i,j) \in I. \tag{17}$$

We will consider the dual problem: we will say that a map f is *feasible* if $f(i) \in J_i$ for $i \in [n]$. We let Ω^* denote the set of feasible f.

$Dual_{map}(W, C, c_0)$:

$$\text{Compute } \max_{\lambda \geq 0} \phi(\lambda, c_0) \text{ where } \phi(\lambda, c_0) = \min_{f \in \Omega^*} \left\{ \sum_{i \in [n]} (W_{i,f(i)} + \lambda C_{i,f(i)}) - \lambda c_0 \right\}.$$

Now it is well known (see for example [7]) that

$$\max_{\lambda \geq 0} \phi(\lambda, c_0) = \min \left\{ \sum_{(i,j)\in I} W_{i,j} x_{i,j} \text{ subject to } (14), (15), (17) \right\}.$$

I.e. maximising ϕ solves the linear program LP_{map}. The basic feasible solutions to the linear program LP_{map} have a rather simple structure. A basis matrix is obtained by replacing a single row of the $n \times n$ identity matrix I_n with coefficients from the LHS of (15) (or it is I_n). Thus, if the associated basic feasible solution is non-integral, then there is a single i^* such that (i) $i \neq i^*$ implies that there is a unique $j(i)$ such that $x_{i,j(i)} = 1$ and $x_{i,j} = 0$ for $j \neq j(i)$ and (ii) there are two indices j_1, j_2 such that $x_{i^*,j_\ell} \neq 0, \ell = 1, 2$.

We are using Corollary 7 to restrict ourselves to feasible f, so that we may use the upper bounds in (12), (13).

Consider the unique (with probability one) basic feasible solution that solves LP_{max}. The optimal shadow price λ^* is also the optimal solution to the dual problem $DUAL_{map}(W, C, c_0)$.

Let the map $f^* = f^*(c_0)$ be obtained from an optimal basic feasible solution to LP_{map} by (i) putting $x_{i^*,j_1} = x_{i^*,j_2} = 0$ and then (ii) choosing j^* to minimise $C_{i^*,j} + \lambda^* W_{i^*,j}$ and then putting $x_{i^*,j^*} = 1$. This yields the map f^*, where $f^*(i) = j(i), i \neq i^*$ and $f^*(i^*) = j^*$.

Let $W_{\max} = \max \left\{W_{i,f^*(i)} : i \in [n]\right\}$ and define $C_{\max}$ similarly. Let W^*_{LP} denote the optimal objective value to LP_{map}. Then we clearly have

$$W(f^*) \leq W^*_{LP} + W_{\max} \text{ and } C(f^*) \leq c_0 + C_{\max}. \tag{18}$$

Lemma 8. *Let $W_{(i,j)}$ and $C_{(i,j)}$, $i, j \leq n$, be i.i.d. random variables uniform on $[0, 1]$. Then f^* is distributed as a random mapping.*

Proof. Fix $f_0 \in [n]^{[n]}$ and a permutation π of $[n]$. The distribution of f^* is invariant with respect to relabelling (permuting) the domain $[n]$, that is $\pi \circ f^*$ and f^* have the same distribution. Therefore,

$$\mathbb{P}\left(f^* = f_0\right) = \mathbb{P}\left(\pi \circ f^* = \pi \circ f_0\right) = \mathbb{P}\left(f^* = \pi \circ f_0\right).$$

□

2.5 Discretisation

We divide the interval $[0, n \log n]$ into n^{10} intervals $[\lambda_i, \lambda_{i+1}]$ of equal length. Then $|\lambda_{i+1} - \lambda_i| \leq n^{-9}$. By standard arguments we have the following claim about the maximum after the discretisation.

Lemma 9. *Almost surely, we have*

$$\max_{\lambda} \phi_{map}(\lambda, c_0) = \max_{i \leq n^{10}} \phi_{map}(\lambda_i, c_0) + O(redn^{-8}). \tag{19}$$

Proof. This follows from a standard argument: we have

$$|\max_{\lambda} \phi_{map}(\lambda, c_0) - \max_{i \leq n^{10}} \phi_{map}(\lambda_i, c_0)| \leq \max_{i \leq n^{10}} \max_{\lambda \in [\lambda_i, \lambda_{i+1}]} |\phi_{map}(\lambda, c_0) - \phi_{map}(\lambda_i, c_0)|$$

and for any λ, λ'

$$|\phi_{map}(\lambda, c_0) - \phi_{map}(\lambda', c_0)| \leq |\min_{T} \sum_{e \in T} (W_e + \lambda C_e) - \min_{F'} \sum_{e \in F'} (W_e + \lambda' C_e)| + |\lambda - \lambda'| c_0.$$

If we take $\tilde{T}$ to be an optimal mapping for λ and $\tilde{T}'$ for λ', we can conclude that

$$\min_{T} \sum_{e \in T} (W_e + \lambda C_e) \leq \sum_{e \in \tilde{T}'} (W_e + \lambda C_e) = \min_{F'} \sum_{e \in F'} (W_e + \lambda' C_e) + \sum_{e \in \tilde{T}'} (\lambda - \lambda') C_e$$

which easily gives (by estimating each C_e by 1 and exchanging the roles of λ and λ')

$$|\min_{T} \sum_{e \in T} (W_e + \lambda C_e) - \min_{F'} \sum_{e \in F'} (W_e + \lambda' C_e)| \leq |\lambda - \lambda'| n.$$

Since $c_0 = O(n)$ and $|\lambda - \lambda_i| \leq n^{-9}$, we finish the argument.

□

The function $\phi_{map}(\lambda, c_0)$ is concave and will be strictly concave with probability one. Let λ^* denote the value of λ maximising ϕ and let λ^{**} be the closest discretised value to λ^*. Let f^{**} be the mapping that minimises $W(f) + \lambda^{**}C(f)$. We will see in the following that

$$\lambda^* \geq \frac{1}{n^2} \text{ w.h.p.} \tag{20}$$

Lemma 10. *Assuming* (20), *then*

$$f^* = f^{**} \text{ } w.h.p.$$

Proof. Consider the dual linear program to LP_{map}. This can be expressed

$$\text{Maximise } \sum_{i=1}^{n} u_i - \lambda c_0 \text{ subject to } \lambda \geq 0,\ u_i - \lambda C_{i,j} \leq W_{i,j}, \text{ for all } i, j.$$

with solution $u_1^*, \ldots, u_n^*, \lambda^*$.

In an optimal basic feasible solution LP_{map}, λ^* will be the optimal shadow price and for a fixed $i \neq i^*$, the reduced cost of the variable $x_{i,j}$ will be $Z_{i,j}^* = W_{i,j} + \lambda^* C_{i,j} - u_i^*$. Because we are considering an optimal basic feasible solution we will have $Z_{i,j}^* \geq 0$ for all i, j and the basic $x_{i,j}$'s will satisfy $Z_{i,j}^* = 0$. It follows from the fact that there is only a single i for which there is no basic $x_{i,j}$, that $f^*(i)$ is chosen to minimise $Z_{i,j}^*$ for at least $n-1$ indices $i \neq i^*$. We have already defined $f^*(i^*)$ to minimise $Z_{i^*,j}$. It only remains to argue that if we replace λ^* by λ^{**} to obtain $Z_{i,j}^{**}$ then w.h.p. the minimising index does not change for any i.

Now $|Z_{i,j}^{**} - Z_{i,j}^*| \leq |\lambda^{**} - \lambda^*| \leq n^{-9}$. Also, if X, Y are independent uniform $[0, 1]$ random variables that $\mathbf{Pr}(X + \lambda Y \in [a, a + \delta]) \leq \delta/\lambda$ for any choice of a, δ, λ. So,

$$\mathbf{Pr}(\exists i : \text{minimiser changes})$$
$$\leq \mathbf{Pr}\left(\exists i, j_1, j_2, k : Z_{i,j,1}^*, Z_{i,j_2}^{**} \in \left[\frac{k}{n^9}, \frac{k+2}{n^9}\right]\right) \leq n^3 n^9 \cdot \left(\frac{2}{\lambda^* n^9}\right)^2 = o(1),$$

under the assumption that $(\lambda^*)^2 n^6 \to \infty$. □

2.6 Cycles

A mapping f gives rise to a digraph $D_F = ([n], \{(v, f(v)) : v \in [n]\}$. The digraph D_F splits into components consisting of directed cycles plus arborescences attached to these cycles.

Lemma 11. *There is a universal constant K such that a uniform random mapping $F : [n] \to [n]$ has at most $K \log n$ cycles with probability at least $1 - O(n^{-50})$.*

Proof. If we condition on the set C of vertices on cycles, then the cycles define a random permutation of the elements of C. One can see this by observing that if we remove the edges from these cycles and replace them with another collection of cycles that cover C then we get another digraph of a mapping. This explains that each set of cycles that covers C has the same set of extensions to a mapping digraph i.e. arises in the same number of mappings.

Let $C = [m]$. Let π be a random permutation of $[m]$. Let X denote the size of the cycle containing 1. Then

$$\mathbb{P}(X = i) = \frac{(m-1)(m-2)\cdots(m-i+1) \times (m-i)!}{m!} = \frac{1}{m}.$$

Explanation: The factor $(m-1)(m-2)\cdots(m-i+1)$ is the number of ways of completing the cycle containing 1 and $(m-i)!$ is the number of ways of computing the vertices not on C.

Now let Y denote the number of cycles in π. From this we can argue that

$$\mathbb{P}(Y \geq t) \leq \mathbb{P}(Bin(t, 1/2) \leq \lceil \log_2 m \rceil).$$

Explanation: We flip a sequence of fair coins. If we get a head in the first one, then we interpret this as vertex 1 being on a cycle C_1 of size at least $m/2$ and then we continue the experiment with $[m] \setminus C_1$. If we get a tail, then we continue the experiment with $[m]$.

So, by the Chernoff bounds, if Z is the number of cycles in a random mapping, then for $K \geq 2$,

$$\begin{aligned}\mathbb{P}(Z \geq K \log_2 n)) &\leq \mathbb{P}(Bin(K \log_2 n, 1/2) \leq \lceil \log_2 n \rceil) \\ &\leq \exp\left\{-\frac{(K-2)^2}{2K^2} \cdot A \log_2 n\right\} = n^{-(K-2)^2/2K}.\end{aligned}$$

□

3 Proof of Theorem 1

It will be convenient to first argue about the cost of an optimal mapping and then amend it to obtain an *almost* optimal arborescence with the (asymptotically) correct cost. Namely, we define $W^*_{map}(c_0)$ to be the optimal value of the integer program IP_{map} of Section 2.4.

First, we show that with high probability

$$W^*_{map}(c_0) \approx \begin{cases} \frac{\pi n}{8c_0}. & \text{Case 1.} \\ f(\beta) - \alpha\beta \text{ where } f'(\beta) = \alpha & \text{Case 2.} \\ (g(\beta) - \alpha\beta)n \text{ where } g'(\beta) = \alpha & \text{Case 3.} \end{cases} \tag{21}$$

and then we modify an *almost* optimal mapping (with the slightly more restricted budget $c_0 - \delta$ for the cost) to obtain an arborescence A which with high probability will satisfy $W(A) \approx W^*_{map}(c_0)$ as well as the cost constraint $C(A) = \sum_{e \in A} C_e \leq c_0$. Since

$$W^*_{arb}(c_0) \geq W^*_{map}(c_0) \approx W(A) \geq W^*_{arb}(c_0),$$

this will show that $W^*_{arb}(c_0) \approx \frac{\pi n}{8c_0}$ in Case 1., etc., as desired.

3.1 A near optimal mapping

Our goal is to show (21). By weak duality or the fact that LP_{map} relaxes IP_{map} we have

$$W^*_{map}(c_0) \geq \max_{\lambda} \phi_{map}(\lambda, c_0). \tag{22}$$

To handle ϕ_{map}, note that the minimum over the mappings is of course attained by choosing the best edge for each vertex, that is

$$\phi_{map}(\lambda, c_0) = \sum_{i \le n} \min_{j \ne i} \{W_{(i,j)} + \lambda C_{(i,j)}\} - \lambda c_0. \tag{23}$$

Now the analysis splits into three cases according to the value of c_0.

Case 1: $c_0 \in \sqrt{\frac{\pi}{8}}[\sqrt{\log n}, n/(\log n)^2]$.

First we take the maximum over i. The function $(1+o(1))\sqrt{\frac{\pi}{2}}\sqrt{\lambda n} - \lambda c_0$ is strictly concave and has a global maximum at $\lambda^* = (1+o(1))\frac{\pi n}{8c_0^2}$, satisfying (20). Note that with our assumption on c_0, this value of λ is in the third range of Lemma 4.

By (4) and the concentration result of Lemma 6 applied to $\varepsilon = n^{-1/5}$, we have

Lemma 12.

$$\phi_{map}(\lambda_i, c_0) = (1+o(1))\sqrt{\frac{\pi}{2}}\sqrt{\lambda_i n} - \lambda_i c_0,$$

for every $i \le n^5$ *with probability at least* $1 - O(n^{-99})$.

Thus the optimal value over $\lambda = \lambda_i$, $i \le n^5$, is

$$\begin{aligned}\max_{i \le n^5} \phi_{map}(\lambda_i, c_0) &= (1+o(1))\sqrt{\frac{\pi}{2}}\sqrt{(\lambda_* + O(n^{-4}))n} - (\lambda_* + O(n^{-4}))c_0 \\ &= (1+o(1))\frac{\pi}{8}\frac{n}{c_0}\end{aligned}$$

which together with Claim 1 gives that with probability at least $1 - O(n^{-99})$

$$\max_{\lambda} \phi_{map}(\lambda, c_0) = (1+o(1))\frac{\pi}{8}\frac{n}{c_0} + O(n^{-3}) = (1+o(1))\frac{\pi}{8}\frac{n}{c_0}. \tag{24}$$

The last step is to tighten the cost constraint a little bit and consider $c_0' = c_0 - 1$. Since $c_0' \approx c_0$, by using (24) twice and recalling (22), we obtain

$$\begin{aligned}W^*_{map}(c_0) \ge \max_{\lambda} \phi_{map}(\lambda, c_0) = (1+o(1))\frac{\pi}{8}\frac{n}{c_0} &= (1+o(1))\frac{\pi}{8}\frac{n}{c_0'} \\ &= (1+o(1))\max_{\lambda} \phi_{map}(\lambda, c_0') \ge W(f^*) - W_{\max},\end{aligned} \tag{25}$$

where $f^* = f^*(c_0')$ is as in (18) and

$$C(f^*) \le c_0' + C_{\max}(f^*) \le c_0' + 1 \le c_0.$$

This means that the solution f^* is feasible and thus $W(f^*) \ge W^*_{map}(c_0)$. We have from Corollary 7 and our expressions for the optimal value of λ that

$$W_{\max} = O\left(1 + \frac{n}{c_0^2}\right)\sqrt{\log n/n} = o\left(\frac{n}{c_0}\right) = o(W(f^*)).$$

Going back to (25) we see that $W^*_{map}(c_0) \approx \frac{\pi}{8}\frac{n}{c_0}$, thus showing (21) holds with probability at least $1 - O(n^{-90})$. Moreover,

$$W^*_{map}(c_0) \approx \max_\lambda \phi_{map}(\lambda, c_0). \tag{26}$$

Case 2: $c_0 = \alpha n$, $\alpha = O(1)$.

If $\alpha > 1/2$ then w.h.p. we can take the mapping $f(v)$ where $W_{(v,f(v))} = \min\left\{W_{(v,w)} : w \neq v\right\}$. Then the sum $\sum_v C_{(v,f(v))}$ being the sum of n independent uniform $[0,1]$ random variables is asymptotically equall to $n/2$ w.h.p. This implies that f defines a feasible mapping w.h.p.

Assume then that $\alpha < 1/2$. We use the argument of Case 1 and we omit details common to both cases. We first check that the optimal value λ^* is in the second range of Lemma 4. To see this observe that if $\lambda = \frac{\beta}{n}$ where $\beta \in \left[\frac{1}{\log n}, \log n\right]$ then $\phi_{map}(\lambda, c_0) \approx f(\beta) - \alpha\beta$. Now Lemma 2 affirms that $f(\beta) - \alpha\beta$ is concave and that there is a unique positive solution β^* to $f'(\beta) = \alpha$. It follows that $\max_\lambda \phi_{map}(\lambda, c_0) \approx f(\beta^*) - \alpha\beta^*$.

We let $c_0' = c_0 - 1 \approx c_0$. Using the continuity of f and $W_{\max} = o(1)$ from (12), we have $W^*_{map}(c_0) \geq (1 + o(1))W(f^*)$ in (25) and by (18) we have $C(f^*) \leq c_0' + 1 = c_0$. Again, (20) is satisfied.

Case 3: $c_0 = \alpha$, $\alpha = O(1)$.

If $\alpha < 1$ then w.h.p. the problem is infeasible. This is because the sum $S = \sum_v \min_w W_{(v,w)}$ is the sum of n i.i.d. random variables and this sum has mean $\frac{n}{n+1}$ and Lemma 6 with $\lambda = 0$ shows that S is concentrated around its mean.

Assume then that $\alpha > 1$. We use the argument of Case 1 and as in Case 2, we omit details common to both cases. We first check that the optimal value λ^* is in the second range of Lemma 4. To see this observe that if $\lambda = \beta n$ where $\beta \in \left[\frac{1}{\log n}, \log n\right]$ then $\phi_{map}(\lambda, c_0) \approx n(g(\beta) - \alpha\beta)$. Now Lemma 3 affirms that $g(\beta) - \alpha\beta$ is concave and that there is a unique positive solution β^* to $g'(\beta) = \alpha$. It follows that $\max_\lambda \phi_{map}(\lambda, c_0) \approx n(g(\beta^*) - \alpha\beta^*)$. It only remains to check that $C_{\max}(f^*) = o(1)$ so that we can apply (18). Again, (20) is satisfied.

We now let $c_0' = c_0 - 1/n^{1/2} \approx c_0$. Using the continuity of g and $W_{\max} \leq 1$ we have $W^*_{map} \geq (1 + o(1))W(f^*)$ in (25) and we have $C(f^*) \leq c_0' + K\frac{\log n}{n} \leq c_0$.

3.2 From a mapping to an arborescence

Case 1:
Fix c_0 and let $c_0' = c_0(1 - \varepsilon)$ with $\varepsilon = n^{-1/4}\log n$. Since $c_0' \approx c_0$, by (21) and (26), we have

$$W^*_{arb}(c_0) \geq W^*_{map}(c_0) \approx \frac{\pi n}{8c_0} \approx \frac{\pi n}{8c_0'} \approx W^*_{map}(c_0') \approx \max_\lambda \phi_{map}(\lambda, c_0').$$

Let the maximum on the right hand side be attained at some λ^* and let λ^{**} be the closest discretized value. Let f^* be as defined in Section 2.4 and f^{**} minimise $W(f) + \lambda^{**}C(f)$. Then, we have from Claim 1 and (18) that

$$
\begin{aligned}
W(f^*) \leq W^*_{map}(c_0) + W_{\max} + O(n^{-3}) \\
C(f^*) \leq c_0' + C_{\max}.
\end{aligned} \tag{27}
$$

We now argue that with high probability it is possible to modify f^* to obtain a feasible arborescence A, that is of cost at most c_0, having weight very close to W^*_{map}.

By Lemmas 8 and 11, with probability at least $1 - O(n^{-10})$, f^* has at most $K \log n$ cycles for some universal constant K. Then the largest component, call it U, has at least $\frac{n}{K \log n}$ vertices.

We consider two cases:

Case 1a: $c_0 \geq n^{1/2}$**:**
For each cycle, choose arbitrarily one vertex belonging to it, say v, remove its out-edge, breaking the cycle and put instead an out-edge connecting it to U. This way $f^*$is transformed into an arborescence, call it A. We have $W^*_{map} = \Omega(n/c_0) = \Omega((\log n)^2)$ and then from (27) and $W_{\max}, C_{\max} \leq 1$ that

$$
\begin{aligned}
W(A) &\leq W(f^*) + K \log n = \left(1 + O\left(\frac{1}{\log n}\right)\right) W^*_{map}. \\
C(A) &\leq c_0' + 1 + O(n^{-3}) + K \log n \leq c_0.
\end{aligned}
$$

Case 1b: $c_0 \leq n^{1/2}$**:**
It follows from $\lambda^* = \Theta(n/c_0^2)$ that $\lambda^* = \Omega(1)$. It then follows from (10) and Lemma 10 that $C_{\max}(f^*) = O(\sqrt{\log n/n})$. If therefore we delete every edge e for which $C_e \geq n^{-1/4}$ from $\vec{K}_n$ and compute an optimal mapping, then w.h.p. we will get the same mapping f^* as without doing the deletion. Now w.h.p., for any vertex v, there are at most $2n^{1/4}$ edges e incident with $C_e < n^{-1/4}$.

Now put back every edge that was deleted and consider the conditional distribution of C_e of a deleted edge e. The distribution of C_e will be uniform $[0, 1]$, conditional on $C_e \geq n^{-1/4}$ and this is uniform $[n^{-1/4}, 1]$. Applying the same transformation from mapping to arborescence as in Case 1, but doing this as cheaply as possible, we see that

$$
\begin{aligned}
&\mathbb{P}\left(\nexists \text{ out-edge } e \in E(v : U) \text{ such that } C_e \in [n^{-1/4}, 2n^{-1/4}]\right) \\
&\qquad \leq \left(1 - \frac{n^{-1/4}}{1 - n^{-1/4}}\right)^{n/K \log n - 2n^{1/4}} \leq e^{-\Omega(n^{3/4}/\log n)}.
\end{aligned}
$$

Taking the union bound over the cycles, we see that with high probability for each cycle there is a choice of an edge with $W_e \leq 1, C_e \leq 2n^{-1/4}$. Thus, the difference of weight between f^* and A is at most $2K \log n$ and the difference of cost is at most $2K \log n \times n^{-1/4}$. Consequently, $C(A) \leq c_0(1 - \varepsilon) + 2Kn^{-1/4} \log n \leq c_0$ and therefore A is feasible and we get

$$
W^*_{arb}(c_0) \leq W(A) \leq \frac{\pi n}{8c_0} + 2K \log n \approx \frac{\pi n}{8c_0}.
$$

This finishes the proof of Case 1.

Case 2:
We have $c_0 = \Omega(n)$ here and $\lambda^{**} = \beta^{**} = \Theta(1)$. We can therefore use (10) to argue that w.h.p.

$\max\{W_{\max}(f^*), C_{\max}(f^*)\} = O(\sqrt{\log n/n})$. We then can proceed as in Case 1b and use edges e such that $W_e, C_e \in [n^{-1/4}, 2n^{-1/4}]$ to transform f^* into an arborescence and w.h.p. change weight and cost by $o(1)$ only.

Case 3:
We have $\lambda^{**} = \beta^{**}n = \Theta(n)$. We can therefore use (10) to argue that w.h.p. $C_{\max}(f^*) = O(\sqrt{\log n/n})$. We proceed as in Case 1b and use edges e such that $W_e \leq 1, C_e \in [n^{-1/4}, 2n^{-1/4}]$ to transform f^* into an arborescence. The extra cost in going from mapping f^* to an arborescence is $O(n^{-1/4}\log n) = o(1)$. The extra weight is $O(\log n)$ which is much smaller than the optimal weight which is $\Omega(n)$ w.h.p.

4 Conclusion

We have determined the asymptotic optimum value to Problem (1) w.h.p. The proof is constructive in that we can w.h.p. get an asymptotically optimal solution (1) by computing arborescence A of the previous section. Our theorem covers almost all of the possibilities for c_0, although there are some small gaps between the 3 cases.

The present result assumes that cost and weight are independent. It would be more reasonable to assume some positive correlation. This could be the subject of future research. One could also consider more than one constraint.

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