Injective Tauberian operators on L_1 and operators with dense range on ℓ_∞ *

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Abstract

There exist injective Tauberian operators on $L_1(0,1)$ that have dense, non closed range. This gives injective, non surjective operators on ℓ_∞ that have dense range. Consequently, there are two quasi-complementary, non complementary subspaces of ℓ_∞ that are isometric to ℓ_∞.

1 Introduction

A (bounded, linear) operator T from a Banach space X into a Banach space Y is called Tauberian provided $T^{**-1}Y = X$. The structure of Tauberian operators when the domain is an L_1 space is well understood and exposed in Gonzáles and Martínez-Abejón's book [5, Chapter 4]. (For convenience they only consider $L_1(\mu)$ when μ is finite and purely nonatomic, but their proofs for the results we mention work for general L_1 spaces.) In particular, [5, Theorem 4.1.3] implies that when X is an L_1 space, an operator $T : X \to Y$ is Tauberian iff whenever (x_n) is a sequence of disjoint unit vectors, there is an

*AMS subject classification: 46E30, 46B08, 47A53 Key words: L_1, Tauberian operator, ℓ_∞

†Supported in part by NSF DMS-1301604 and U.S.-Israel Binational Science Foundation

‡Supported in part by U.S.-Israel Binational Science Foundation. Participant NSF Workshop in Analysis and Probability, Texas A&M University

§T. Tkocz thanks his PhD supervisor, Keith Ball, for his invaluable constant advice and encouragement

N so that the restriction of T to $[x_N]_{n=N}^\infty$ is an isomorphism (and, moreover, the norm of the inverse of the restricted operator is bounded independently of the disjoint sequence). From this it follows that an injective operator $T : X \to Y$ is Tauberian iff it isomorphically preserves isometric copies of ℓ_1 in the sense that the restriction of T to any subspace of X that is isometrically isomorphic to ℓ_1 is an isomorphism. (Recall that a subspace of an L_1 space is isometrically isomorphic to ℓ_1 iff it is the closed linear span of a sequence of non zero disjoint vectors [11, Chapter 14.5].) Since Tu is Tauberian if T is Tauberian and u is an isomorphism, one deduces that an injective Tauberian operator from an L_1 space isomorphically preserves isomorphic copies of ℓ_1 in the sense that the restriction of T to any subspace of X that is isomorphic to ℓ_1 is an isomorphism. Thus injective Tauberian operators from an L_1 space are opposite to ℓ_1-singular operators; i.e., operators whose restriction to every subspace isomorphic to ℓ_1 is *not* an isomorphism.

The main result in this paper is a negative solution to [5, Problem 1]: Suppose T is a Tauberian operator on an L_1 space. Must T be upper semi-Fredholm; i.e., must the range $\mathcal{R}(T)$ of T be closed and the null space $\mathcal{N}(T)$ of T be finite dimensional? The basic example is a Tauberian operator on $L_1(0,1)$ that has infinite dimensional null space. This is rather striking because the Tauberian condition is equivalent to the statement that there is $c > 0$ so that the restriction of the operator to $L_1(A)$ is an isomorphism whenever the subset A of $[0,1]$ has Lebesgue measure at most c.

In fact, we show that there is an injective, dense range, non surjective Tauberian operator on $L_1(0,1)$. Since T is Tauberian, T^{**} is also injective, so $\mathcal{R}(T^*)$ is dense and proper, and T^* is injective because $\mathcal{R}(T)$ is dense. This solves a problem [10] the second author raised on MathOverFlow.net that led to the collaboration of the authors.

2 The examples

We begin with a lemma that is an easy consequence of characterizations of Tauberian operators on L_1 spaces.

Lemma 1 *Let X be an L_1 space and T an operator from X to a Banach space Y. The operator T is Tauberian if and only if there is $r > 0$ and a natural number N so that if $(x_n)_{n=1}^N$ are disjoint unit vectors in X, then $\max_{1\le n\le N} \|Tx_n\| \ge r$.*

Proof: The condition in the lemma clearly implies that if (x_n) is a disjoint sequence of unit vectors in X, then $\liminf_n \|Tx_n\| > 0$, which is one of the equivalent conditions for T to be Tauberian [5, Theorem 4.1.3]. On the other hand, suppose that there are disjoint collections $(x_k^n)_{k=1}^n$, $n = 1, 2, \dots$ with $\max_{1\le k\le n} \|Tx_k^n\| \to 0$ as $n \to \infty$. Then the closed sublattice generated by $\cup_{n=1}^\infty (x_k^n)_{k=1}^n$ is a separable abstract L_1 space (meaning that it is a Banach lattice such that $\|x + y\| = \|x\| + \|y\|$ whenever $|x| \vee |y| = 0$) and hence is order isometric to $L_1(\mu)$ for some probability μ by Kakutani's theorem (see e.g. [7, Theorem 1.b.2]). Choose $1 \le k(n) \le n$ so that the support of $x_{k(n)}^n$ in $L_1(\mu)$ has measure at most $1/n$. Since T is Tauberian, by [5, Proposition 4.1.8] necessarily $\liminf_n \|Tx_{k(n)}^n\| > 0$, which is a contradiction. ∎

The reason that Lemma 1 is useful for us is that the condition in the Lemma is stable under ultraproducts. Call an operator that satisfies the condition in Lemma 1 (r, N)-Tauberian. For background on ultraproducts of Banach spaces and of operators, see [4, Chapter 8]. We use the fact that the ultraproduct of L_1 spaces is an abstract L_1 space and hence is order isometric to $L_1(\mu)$ for some measure μ.

Lemma 2 *Let (X_k) be a sequence of L_1 spaces, and for each k let T_k be a norm one linear operator from X_k into a Banach space Y_k. Assume that there is $r > 0$ and a natural number N so that each operator T_k is (r, N)-Tauberian. Let $\mathcal{U}$ be a free ultrafilter on the natural numbers. Then $(T_k)_\mathcal{U} : (X_k)_\mathcal{U} \to (Y_k)_\mathcal{U}$ is (r, N)-Tauberian.*

Here $(T_k)_\mathcal{U}$ is the usual ultraproduct of the sequence (T_k), defined by

$$(T_k)_\mathcal{U}(x_k) = (T_k x_k).$$

Proof: The vectors (x_k) and (y_k) are disjoint in the abstract L_1 space $(X_k)_\mathcal{U}$ iff $\lim_\mathcal{U} \||x_k| \wedge |y_k|\| = 0$, so it is only a matter of proving that if T is (r, N)-Tauberian from some L_1 space X, then for each $\varepsilon > 0$ there is $\delta > 0$ so that if $x_1, \dots, x_N$ are unit vectors in X and $\||x_n| \wedge |x_m|\| < \delta$ for $1 \le n < m \le N$, then $\max_{1\le n\le N} \|Tx_n\| > r - \varepsilon$. But if $x_1, \dots, x_N$ are unit vectors that are ε-disjoint as above, and $y_1, \dots, y_n$ are defined by

$$y_n := [|x_n| - (|x_n| \wedge (\vee\{|x_m| : m \neq n\})]\mathrm{sign}(x_n),$$

then the y_n are disjoint and all have norm at least $1 - N\delta$. Normalize the y_n and apply the (r, N)-Tauberian condition to this normalized disjoint sequence to see that $\max_{1 \le n \le N} \|Tx_n\| > r - \varepsilon$ if $\delta = \delta(\varepsilon, N)$ is sufficiently small. ∎

An example that answers [5, Problem 1] is the restriction of an ultraproduct of operators on finite dimensional L_1 spaces constructed in [3].

Theorem 1 *There is a Tauberian operator T on $L_1(0,1)$ that has an infinite dimensional null space. Consequently, T is not upper semi-Fredholm.*

Proof: An immediate consequence of [3, Proposition 6 & Theorem 1] is that there is $r > 0$ and a natural number N so that for all sufficiently large n there is a norm one (r, N)-Tauberian operator T_n from ℓ_1^n into itself with $\dim \mathcal{N}(T_n) > rn$. The ultraproduct $\tilde{T} := (T_n)_{\mathcal{U}}$ is then a norm one (r, N)-Tauberian operator on the gigantic L_1 space $X_1 := (\ell_1^n)_{\mathcal{U}}$, and the null space of $\tilde{T}$ is infinite dimensional. Take any separable infinite dimensional subspace X_0 of $\mathcal{N}(\tilde{T})$ and let X be the closed sublattice of X_1 generated by X_0. Let Y be the sublattice of X_1 generated by $\tilde{T}X$ and let T be the restriction of $\tilde{T}$ to X, considered as an operator into Y. So X and Y are separable L_1 spaces and by Lemmas 1 and 2 the operator T is Tauberian. Of course, by construction $\mathcal{N}(T)$ is infinite dimensional and reflexive (because T is Tauberian). Thus X is not isomorphic to ℓ_1 and hence is isomorphic to $L_1(0,1)$. So is Y, but that does not matter: Y, being a separable L_1 space, embeds isometrically into $L_1(0,1)$. ∎

We want to “soup up” the operator T in Theorem 1 to get an injective, non surjective, dense range Tauberian operator on $L_1(0,1)$. We could quote a general result [6, Theorem 3.4] of González and Onieva to shorten the presentation, but we prefer to give a short direct proof.

We recall a simple known lemma:

Lemma 3 *Let X and Y be separable infinite dimensional Banach spaces and $\varepsilon > 0$. Let Y_0 be a countable dimensional dense subspace of Y. Then there is a nuclear operator $u : X \to Y$ so that u is injective and $\|u\|_\wedge < \varepsilon$ and $uX \supset Y_0$.*

Proof: Recall that an M-basis for a Banach space X is a biorthogonal system $(x_\alpha, x_\alpha^*) \subset X \times X^*$ such that the linear span of (x_α) is dense in X and $\cap_\alpha \mathcal{N}(x_\alpha^*) = \{0\}$. Every separable Banach space X has an M-basis

[8]; moreover, the vectors (x_α) in the M-basis can span any given countable dimensional dense subspace of X.

Take M-bases (x_n, x_n^*) and (y_n, y_n^*) for X and Y, respectively, normalized so that $\|x_n^*\| = 1 = \|y_n\|$ and such that the linear span of (y_n) is Y_0. Choose $\lambda_n > 0$ so that $\sum_n \lambda_n < \varepsilon$ and set $u(x) = \sum_n \lambda_n \langle x_n^*, x \rangle y_n$. ∎

Theorem 2 *There is an injective, non surjective, dense range Tauberian operator on* $L_1(0,1)$.

Proof: By Theorem 1 there is a Tauberian operator T on $L_1(0,1)$ that has an infinite dimensional null space. By Lemma 3 there is a nuclear operator $\tilde{v} : \mathcal{N}(T) \to L_1(0,1)$ that is injective and has dense range, and we can extend $\tilde{v}$ to a nuclear operator v on $L_1(0,1)$. We can choose $\tilde{v}$ so that $\tilde{v}(\mathcal{N}(T)) \cap T L_1(0,1)$ is infinite dimensional by the last statement in Lemma 3. This guarantees that the Tauberian operator $T_1 := T + v$ has an infinite dimensional null space (this allows us to avoid breaking the following argument into cases).

Now $\mathcal{N}(T_1) \cap \mathcal{N}(T) = \{0\}$, so again by Lemma 3 and the extension property of nuclear operators there is a nuclear operator $u : L_1(0,1)/\mathcal{N}(T) \to \ell_1$ so that the restriction of u to $Q_{\mathcal{N}(T)} \mathcal{N}(T_1)$ is injective and has dense range (here for a subspace E of X, the operator Q_E is the quotient mapping from X onto X/E). Finally, define $T_2 : L_1(0,1) \to L_1(0,1) \oplus_1 \ell_1$ by $T_2 x := T_1 x \oplus u Q_{\mathcal{N}(T)} x$. Then T_2 is an injective Tauberian operator with dense range. T_2 is not surjective because $P_{\ell_1} T_2$ is nuclear by construction, where P_{ℓ_1} is the projection of $L_1(0,1) \oplus_1 \ell_1$ onto $\{0\} \oplus_1 \ell_1$. Since $L_1(0,1) \oplus_1 \ell_1$ is isomorphic to $L_1(0,1)$, this completes the proof. ∎

Corollary 1 *There is an injective, dense range, non surjective operator on* ℓ_∞. *Consequently, there is a quasi-complementary, non complementary decomposition of* ℓ_∞ *into two subspaces each of which is isometrically isomorphic to* ℓ_∞.

Proof: Let T be an injective, dense range, non surjective Tauberian operator on $L_1(0,1)$ (Theorem 2). Since T is Tauberian, T^{**} is also injective, so T^* has dense range but T^* is not surjective because its range is not closed, and T^* is injective because T has dense range. The operator T^* translates to an operator on ℓ_∞ that has the same properties because L_∞ is isomorphic to ℓ_∞ by an old result due to Pełczyński (see, e.g., [1, Theorem 4.3.10]) (notice however that, unlike T^*, the operator on ℓ_∞ cannot be weak* continuous).

For the "consequently" statement, let S be any norm one injective, dense range, non surjective operator on ℓ_∞. In the space $\ell_\infty \oplus_\infty \ell_\infty$, which is isometric to ℓ_∞, define $X := \ell_\infty \oplus \{0\}$ and $Y := \{(x, Sx) : x \in \ell_\infty\}$. Obviously X and Y are isometric to ℓ_∞ and $X + Y = \ell_\infty \oplus S\ell_\infty$, which is a dense proper subspace of $\ell_\infty \oplus_\infty \ell_\infty$. Finally, $X \cap Y = \{0\}$ since S is injective, so X and Y are quasi-complementary, non complementary subspaces of $\ell_\infty \oplus_\infty \ell_\infty$. ■

Theorem 2 and the MathOverFlow question [10] suggest the following problem: Suppose X is a separable Banach space (so that X^* is isometric to a weak* closed subspace of ℓ_∞) and X^* is non separable. Is there a dense range operator on X^* that is not surjective? The answer is "no": Argyros, Arvanitakis, and Tolias [2] constructed a separable space X so that X^* is non separable, hereditarily indecomposable (HI), and every strictly singular operator on X^* is weakly compact. Since X^* is HI, every operator on X^* is of the form $\lambda I + S$ with S strictly singular. If $\lambda \neq 0$, then $\lambda I + S$ is Fredholm of index zero by Kato's classical perturbation theory. On the other hand, since every weakly compact subset of the dual to a separable space is norm separable, every strictly singular operator on X^* has separable range. (Thanks to Spiros Argyros for bringing this example to our attention.)

Any operator T on l^∞ that has dense range but is not surjective has the property that 0 is an interior point of $\sigma(T)$. This follows from Thm 2.6 in [9], where it is shown that $\partial\sigma(T) \subset \sigma_p(T^*)$ for any operator T acting on a $C(K)$ space which has the Grothendieck property.

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